Bifurcation and Symmetry in Convection

By

James William Swift

A.B. (University of California, Santa Cruz) 1978

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Physics

in the

GRADUATE DIVISION

OF THE

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved: Edgar Knobloch 12/10/84

Chairman Date

Allan N. Kaufman 12/10/84

Jerrold E. Marsden 12/6/84

Bifurcation and Symmetry in Convection

James W. Swift

Abstract

The theory of bifurcation with symmetry is applied to the onset of convection in a fluid layer heated from below. Doubly diffusive convection illustrates the general theory, which describes selection between the possible cellular patterns: rolls, hexagons, triangles, squares, and rectangles. Double periodicity in the horizontal plane is imposed, thus allowing only a finite number of convection rolls to become unstable at the onset of convection. Each roll has a time-dependent complex amplitude and the center manifold theorem allows a complete description of the dynamics near the instability in terms of an ordinary differential equation for the critical amplitudes. These ordinary differential equations are called *normal forms*. For the cases discussed here there are 1, 2, 3, 4, or 6 complex amplitudes (i.e. rolls) which go unstable simultaneously. The normal forms have a high degree of symmetry which allows a complete characterization of the dynamics in terms of a few parameters which cannot be eliminated through scaling. These parameters are evaluated for doubly diffusive convection. A classification of all possible generic bifurcations is given for the simplest realization of each type of double periodicity. Some degenerate bifurcations and their unfoldings are classified. Since the classification does not rely on the details of the problem, this work is relevant to any bifurcation problem with the spatial symmetry of the plane when the instability has a preferred wavelength which is neither zero nor infinity.

Chapter Two

Bifurcation Theory and Normal Forms

This chapter describes the bifurcations which can occur in convection when the horizontal planforms are doubly periodic. For the results of this chapter to hold, all that is required of a physical problem is that it be symmetric with respect to rigid motions in a two-dimensional plane, that the linear stability of the conduction solution has a real eigenvalue go through zero, and that the wavelength of the most unstable disturbance is neither zero nor infinity. The normal forms relevant to Hopf bifurcations, where the instability is due to a complex conjugate pair of eigenvalues crossing into the right half plane, are discussed in the next chapter.

The results of this chapter are based on the center manifold theorem. This theorem allows the partial differential equations of convection to be reduced, in certain cases, to a few ordinary differential equations. The chapter starts with a review of bifurcation theory, and the simplest bifurcations are introduced. An understanding of these simple bifurcations is a prerequisite for what follows.

In section 2.3, an example of bifurcation with symmetry is discussed. This example displays the essential behavior of many of the bifurcations which occur in convection. Then, the least degenerate (simplest) bifurcations of convection on a square or rhombic lattice are classified, using the correspondence to the example studied earlier.

Convection on a hexagonal lattice must be treated separately from convection on the other lattices. The Boussinesq approximation plays an important role in pattern selection on a hexagonal lattice. Four different normal forms are appropriate, depending on the degree to which the Boussinesq symmetry is valid.

The chapter ends with a discussion of the lattice function, which displays the results of all the lattices.

An intuitive, physical approach to the theory is used whenever possible, rather than sophisticated mathematics.

2.1. A Quick Review of Bifurcation Theory

Bifurcation theory is the study of the branching of solutions of ordinary and partial differential equations. This branching is always accompanied by a change in the stability of the solutions. The stability properties of a stationary solution are found by linearizing the equations about the fixed point. This linearization is a linear operator. If the system is an ordinary differential equation, the linear operator can be represented by a matrix. The eigenvalues of this linear operator (or matrix) determine the linear stability. For the discussion below, assume that the system is an ordinary differential equation, hereafter referred to as an ODE. Define $\mathbf{X}_j$ and λ_j as the eigenvectors and corresponding eigenvalues of the matrix. (In the case of partial differential equations, the eigenvectors are replaced by eigenfunctions.) The general solution of the linearized equations is a linear superposition of

$$\mathbf{X}_j e^{\lambda_j t} . \qquad (2\text{-}1)$$

If all the eigenvalues have negative real part then any perturbation decays and the fixed point is stable (to small enough perturbations). If any eigenvalue has a positive real part, then a perturbation along the corresponding eigenvector will grow exponentially, and the fixed point is unstable. If an eigenvalue has zero real part, then the nonlinear terms determine whether a perturbation along the corresponding eigenvector grows or decays.

The linear space defined by the set of eigenvectors whose eigenvalues have negative real part is called the *stable eigenspace*. Likewise, the *center* *(unstable) eigenspace* is spanned by the eigenvectors corresponding to eigenvalues with zero (positive) real part.

The center manifold is an example of an *invariant manifold*: a subspace of the phase space which is invariant under the dynamics. The stable, unstable, and center manifolds are tangent to the stable, unstable, and center eigenspaces of a fixed point, respectively. It is not obvious that such invariant manifolds exist. The *center manifold theorem* states that such an invariant manifold does indeed exist. There are analogous theorems for the existence of the other invariant manifolds (see Hirsch *et al.* 1967). The book by Marsden & McCracken (1976) describes how the center manifold theorem is used in bifurcation theory. The following statement of the center manifold theorem is taken from Marsden & McCracken (1976, p. 47).

THEOREM: *Let Z be a smooth Banach space and let F_t be a C^0 semiflow defined in a neighborhood of $O \in Z$ for $0 \le t \le \tau$. Assume $F_t(O) = O$ and that for $t > 0$, $F_t(x)$ is C^{k+1} jointly in t and x. Assume that the spectrum of the linear semigroup $DF_t(O): Z \to Z$ is of the form $e^{t(\sigma_1 \cup \sigma_2)}$ where σ_1 lies on the imaginary axis and σ_2 lies in the left half plane $\mathrm{Re}(\sigma_2) < -\sigma < 0$. Let Y be the generalized eigenspace corresponding to the part of the spectrum on the unit circle. Assume $\dim Y = d < \infty$.*

Then there exists a neighborhood V of O in Z and a C^k submanifold $M \subset V$ of dimension d passing through O and tangent to Y at O such that

(a) (Local invariance): *If $x \in M$, $t > 0$, and $F_t(x) \in V$, then $F_t(x) \in M$.*

(b) (Local attractivity): *If $t > 0$ and $F_t^n(x)$ remains defined and in V for all $n = 0, 1, 2, \cdots$, then $F_t^n(x) \to M$ as $n \to \infty$.*

For what follows, it is not necessary to understand the details of this theorem. However, a few remarks are in order:

• The Banach space formulation is general enough to apply to the partial different equations of convection.

• The correspondence between the notation in the theorem and that used here is:

$$\begin{aligned} &\sigma_1 = \{\lambda_j \mid \mathrm{Re}(\lambda_j) = 0\}; \\ &\sigma_2 = \{\lambda_j \mid \mathrm{Re}(\lambda_j) < -\sigma < 0\}; \\ &Y = \mathrm{span}\{X_j \mid \mathrm{Re}(\lambda_j) = 0\} \text{ is the center eigenspace;} \\ &M \text{ is the center manifold.} \end{aligned} \tag{2-2}$$

• The dimension of the center eigenspace must be less than infinity, and the stable part of the spectrum (σ_2) must be bounded away from zero. Both of these conditions are violated in convection, *unless* double periodicity is imposed.

Rather than present a rigorous mathematical treatment of bifurcation theory, an example is used to illustrate the ideas. Consider the following system of ODEs in the plane:

$$\dot{x} = \lambda x - x^3 + xy \tag{2-3}$$

$$\dot{y} = -y + ax^2, \tag{2-4}$$

where $(x, y) \in \mathbb{R}^2$, and λ and a are real, fixed parameters.

Note that this ODE is symmetric under the reflection through the y axis.

$$(x, y) \to (-x, y). \tag{2-5}$$

The point $x = y = 0$ is a stationary solution for all values of λ, and the linearization is

$$\dot{x} = \lambda x \tag{2-6}$$

$$\dot{y} = -y \tag{2-7}$$

When $\lambda < 0$ the stable eigenspace is the whole $x - y$ plane. When $\lambda > 0$ the stable eigenspace is the y axis and the unstable eigenspace is the x axis. At precisely $\lambda = 0$ the x axis is the center eigenspace.

The stable and unstable manifolds are both one-dimensional when $\lambda > 0$. These manifolds are invariant under the dynamics, and tangent to the stable and unstable eigenspaces. Note that the y axis, defined by $x = 0$, is invariant under the dynamics, since $\dot{x} = 0$ when $x = 0$. This result is forced by the symmetry (2-5); since the y axis is invariant under the reflection it is also invariant under the dynamics. The y axis is therefore the stable manifold.

The unstable manifold can be written as a Taylor expansion,

$$y = \alpha x^2 + O(x^3), \tag{2-8}$$

where α must be determined. The fact that the unstable manifold is invariant under the flow implies that

$$\dot{y} = 2\alpha x\dot{x} + O(x^3). \tag{2-9}$$

When the ODE (2-4) is substituted into this equation, one finds

$$-y + ax^2 = 2\alpha\lambda x^2 + O(x^3), \tag{2-10}$$

which becomes

$$(-\alpha + a)x^2 = 2\alpha\lambda x^2 + O(x^3) \tag{2-11}$$

when equation (2-8) is substituted for y. The above equation determines α:

$$\alpha = \frac{a}{(1+2\lambda)}. \tag{2-12}$$

The unstable manifold, which exists for $\lambda > 0$, is therefore

$$y = \frac{a}{(1+2\lambda)} x^2 + O(x^3). \tag{2-13}$$

When $\lambda = 0$, the center manifold can be found by the same procedure:

$$y = ax^2 + O(x^3). \tag{2-14}$$

It is natural to use x as the coordinate of the center manifold. This is done by projecting the center manifold onto the x axis. The dynamics *on* the center manifold are given by inserting (2-14) into equation (2-3):

$$\begin{aligned} \dot{x} &= -x^3 + x\left[ax^2 + O(x^3)\right] \\ &= (a-1)x^3 + O(x^4). \end{aligned} \tag{2-15}$$

Therefore the sign of $(a-1)$ determines the stability of the origin. This stability (or instability) is very sensitive to perturbations in the equations, since any linear terms will dominate the cubic terms when x is small enough.

A fixed point is called ***hyperbolic*** if it has no eigenvalues with zero real part. The behavior of a hyperbolic fixed point is not sensitive to perturbations in the equations. A theorem due to Hartman (1973) says that there is a (nondifferentiable) change of coordinates which eliminates all the nonlinear terms in the neighborhood of a hyperbolic fixed point. When λ is fixed and nonzero, the qualitative behavior of the nonlinear ODE (2-6), (2-7) near $x = y = 0$ is the same as the linearization (2-3), (2-4). When λ is small, however, the neighborhood of $x = y = 0$ described in the theorem is also small.

In order to capture the transition from negative to positive λ, the system is extended to include λ as an independent variable, treated equally with with x and y. The resulting three-dimensional system of ODEs is

$$\dot{x} = \lambda x - x^3 + xy \tag{2-16}$$

$$\dot{y} = -y + ax^2 \tag{2-17}$$

$$\dot{\lambda} = 0. \tag{2-18}$$

Now the center manifold is ***two-dimensional*** when $\lambda = 0$, and one-dimensional otherwise. The center manifold at $\lambda = 0$ has coordinates (x, λ). The dynamics on the center manifold are

$$\dot{x} = \lambda x + (a-1)x^3 + O(x^5) + O(\lambda x^3) + O(\lambda^2 x) \tag{2-19}$$

$$\dot{\lambda} = 0. \tag{2-20}$$

Note that the coefficient of the cubic term can be calculated *at* $\lambda = 0$. This simplifies the calculations.

Assuming $a \neq 1$, the variable x can be scaled by

$$x \to \frac{x}{\sqrt{|a-1|}} \tag{2-21}$$

and the system can be truncated to give the ***normal form*** for the ***pitchfork bifurcation***:

$$\dot{x} = \lambda x + x^3 \text{ when } a-1>0, \text{ and} \tag{2-22}$$

$$\dot{x} = \lambda x - x^3 \text{ when } a-1<0. \tag{2-23}$$

The pitchfork bifurcation is forced by the $x \to -x$ reflectional symmetry. The equation for $\dot{x}$ on the center manifold must have only odd order terms in x. The bifurcation is a pitchfork provided the coefficient of x^3 is nonzero.

In order to deserve the designation as a normal form, it must be demonstrated that the qualitative features of equation (2-22) or (2-23) are unchanged by adding higher order terms, such as

$$O(x^5),\ O(\lambda x^3),\ \text{and } O(\lambda^2 x). \tag{2-24}$$

The higher order terms do not change the qualitative features provided all of the fixed points are hyperbolic when $\lambda \neq 0$. The analysis of the normal form shows that the fixed points are indeed hyperbolic. In addition to the stationary solution at $x=0$, there is a fixed point at

$$\lambda \pm x^2 = 0. \tag{2-25}$$

The upper sign is for equation (2-22), and the lower sign for equation (2-23). The linearization of the ODE about the new fixed point is

$$\left.\frac{\partial \dot{x}}{\partial x}\right|_{\lambda \pm x^2 = 0} = \pm 2x^2 \tag{2-26}$$

Therefore in the "+" version of the normal form (2-22), the nonzero solutions are unstable, and exist for $\lambda < 0$. This is called a ***subcritical bifurcation***. Conversely, the normal form (2-23) corresponds to a ***supercritical bifurcation***, where the nonzero solutions are stable and exist when $\lambda > 0$.

It is a general feature that subcritical solutions, i.e. those coexisting with a stable solution at the origin, are unstable. On the other hand, supercritical solutions have a stable eigenvector pointing in the direction towards the origin. Fig. 2-1 gives a pictorial description of the two bifurcations.

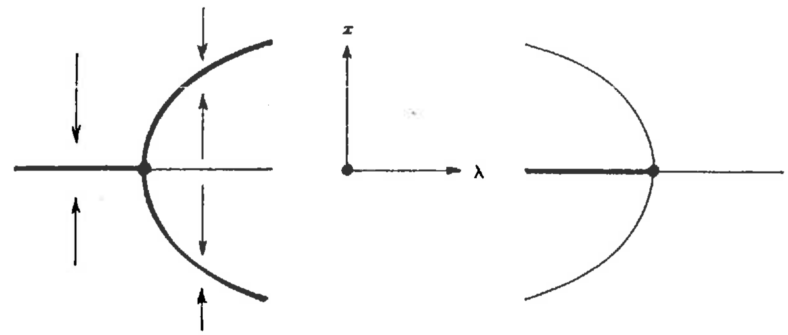

Fig. 2-1. The ***bifurcation diagrams*** for the two cases of the ***pitchfork bifurcation***: (a) the supercritical bifurcation, equation (2-23), and (c) the subcritical bifurcation, equation (2-22). These diagrams plot the solutions as a function of the bifurcation parameter λ. The stable solutions are indicated by thick lines, and the unstable solutions by thin lines. The axes of (a) and (c) are shown in fig. (b). A few trajectories of the two-dimensional system in x and λ, equations (2-19) and (2-20), are drawn in fig. (a).

The ***Hopf bifurcation*** is closely related to the pitchfork bifurcation. It occurs when a complex conjugate pair of eigenvalues $(\lambda \pm i\omega)$ crosses into the right half plane. The normal form for the Hopf bifurcation is

$$\dot{z} = (\lambda + i\omega)z + az\,|z|^2, \tag{2-27}$$

where the z and a are complex, and λ and ω are real. This normal form can be reduced to the pitchfork by writing z in polar coordinates,

$$z = re^{i\varphi}. \tag{2-28}$$

The time derivatives of z and $\bar{z}$ are

$$\dot{z} = \dot{r}e^{i\varphi} + i\dot{\varphi}re^{i\varphi}, \text{ and } \dot{\bar{z}} = \dot{r}e^{-i\varphi} - i\dot{\varphi}re^{-i\varphi}. \tag{2-29}$$

The time derivatives of r and φ can be isolated to give