MAT 136 (Calculus I) Prof. Swift

In-class worksheet: Solving Initial Value Problem, using the FToC II

Recall this problem from the WeBWorK set on Indefinite Integrals:
"Consider the function $f(x) = x^2 - 1$. Let $F(x)$ be the antiderivative of $f(x)$ that satisfies $F(1) = 3$. Then $F(x) =$ ______." This is the same question as "Solve the Initial Value Problem $\frac{dy}{dx} = x^2 - 1$, $y(1) = 3$."

Solution: $F(x) = \frac{x^3}{3} - x + C$ is the general antiderivative. Solve $F(1) = 3$ for C. This is $\frac{1}{3} - 1 + C = 3$, or $C = 3 - (\frac{1}{3} - 1) = \frac{11}{3}$. So the solution is $F(x) = \frac{x^3}{3} - x + \frac{11}{3}$

The Fundamental Theorem of Calculus II allows us to write the solution as

$$y = F(x) = 3 + \int_1^x (t^2 - 1)dt$$

since $F'(x) = 0 + \frac{d}{dx}\int_0^x (t^2 - 1)dt = x^2 - 1$, and $F(1) = 3 + \int_1^1 (t^2 - 1)dt = 3 + 0 = 3$.

1. Evaluate the integral in the displayed equation to get an explicit formula for $F(x)$.

$F(x) = 3 + \int_1^x (t^2-1)dt = 3 + (\frac{t^3}{3} - t)\Big|_1^x = 3 + \frac{x^3}{3} - x - (\frac{1^3}{3} - 1) = \frac{x^3}{3} - x + 3 - \frac{1}{3} + 1 = \boxed{\frac{x^3}{3} - x + \frac{11}{3}}$

Theorem: The solution to the Initial Value Problem (IVP) $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$ is $y = y_0 + \int_{x_0}^x f(t)\, dt$. The dummy variable t can be replaced by any variable, *including* x, to give a formula that you may prefer:

$$y = y_0 + \int_{x_0}^x f(x)\, dx$$

2. Solve the IVP $\frac{dy}{dx} = x^2 - 1$, $y(1) = 3$ using this new formula.

$y = 3 + \int_1^x (x^2-1)dx = 3 + (\frac{x^3}{3} - x)\Big|_1^x = 3 + \frac{x^3}{3} - x - (\frac{1^3}{3} - 1)$

$= 3 + \frac{x^3}{3} - x + \frac{2}{3} = \boxed{\frac{x^3}{3} - x + \frac{11}{3}}$

3. Solve the IVP $\frac{dy}{dx} = \sin(x)$, $y(0) = 0$.

$y = 0 + \int_0^x \sin(x)dx = -\cos(x)\Big|_0^x$

$= -\cos(x) - (-\cos(0))$

$= -\cos(x) - (-1)$

$\boxed{y = 1 - \cos(x)}$