MAT 136 (Calculus I) Prof. Swift

In-class worksheet: The shape of graphs

As a reminder, please do not *think, say or write* "it's positive so it's increasing" when you mean "f' is positive so f is increasing".

Let the function f be defined by $f(x) = e^{-x^2/2}$ for this entire worksheet.

0. Does the graph of f have any x-intercepts? No. $e^{-x^2/2} > 0$ for all x.

In general, one should ~~always~~ usually compute the y-intercept.

$f(0) = e^{-0^2/2} = e^0 = 1$. So $\boxed{(0,1) \text{ is a point on the graph.}}$

1. Compute $f'(x)$ and find the largest interval(s) on which f is increasing, and on which f is decreasing.

$$f'(x) = e^{-x^2/2} \cdot \frac{d}{dx}(-x^2/2) = -x e^{-\frac{x^2}{2}} = \begin{cases} >0 & \text{for } x<0 \\ =0 & \text{for } x=0 \\ <0 & \text{for } x>0 \end{cases}$$

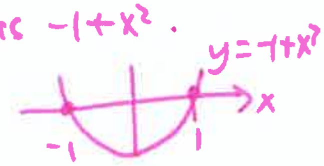
So, $\boxed{\begin{array}{l} f \text{ is increasing on } (-\infty, 0], \\ f \text{ is decreasing on } [0, \infty). \end{array}}$

(Proof: $f'(x) \geq 0$ on $(-\infty, 0]$, and only $x=0$ satisfies $f'(x) = 0$.)

2. Compute $f''(x)$ and simplify your result. Find the largest interval(s) on which f is concave up, and on which f is concave down.

$$f''(x) = -1 \cdot e^{-\frac{x^2}{2}} + (-x)\, e^{-x^2/2}(-x) = -e^{-x^2/2} + x^2 e^{-x^2/2}$$

$\boxed{f''(x) = (-1+x^2)\, e^{-x^2/2}}$ $f''(x)$ has the same sign as $-1+x^2$.



$$f''(x) \begin{cases} >0 & \text{for } x<-1 \\ =0 & \text{for } x=-1 \\ <0 & \text{for } -1<x<1 \\ =0 & \text{for } x=1 \\ >0 & \text{for } x>1 \end{cases}$$

Thus,

$\boxed{\begin{array}{l} f \text{ is concave up on } (-\infty, -1] \text{ and on } [1, \infty). \\ f \text{ is concave down on } [-1, 1]. \end{array}}$

3. Evaluate f at all the numbers that are endpoints of the intervals you found in questions 1 and 2. This will give you three points on the graph $y = f(x)$. Give exact answers, and decimal approximations using a calculator or your phone.

The endpoints are -1, 0, and 1.

$f(-1) = e^{-(-1)^2/2} = e^{-1/2} = 0.61\ldots$, $f(0) = e^0 = 1$, $f(1) = e^{-\frac{1}{2}}$

The 3 points on graph of f are $(-1, e^{-1/2})$, $(0,1)$, and $(1, e^{-\frac{1}{2}})$.

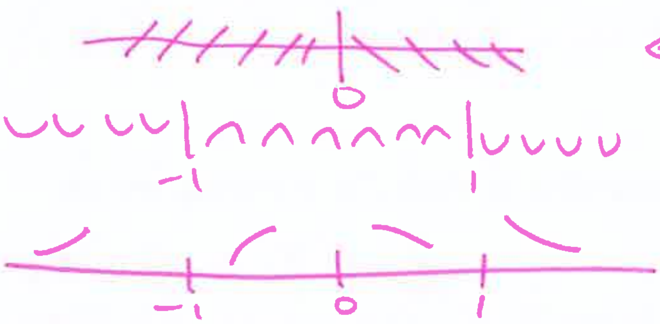
4. Write four sentences following the pattern below. In each sentence, fill in the first blank with an interval, fill in the second blank with “increasing” or “decreasing”, and fill in the third blank with “concave up” or “concave down”.

On the interval $(-\infty, -1]$, the function f is increasing and concave up.

On the interval $[-1, 0]$, the function f is increasing and concave down.

On the interval $[0, 1]$, the function f is decreasing and concave down

On the interval $[1, \infty)$, the function f is decreasing and concave up



← f is increasing or decreasing here.

← f is concave up or down here.

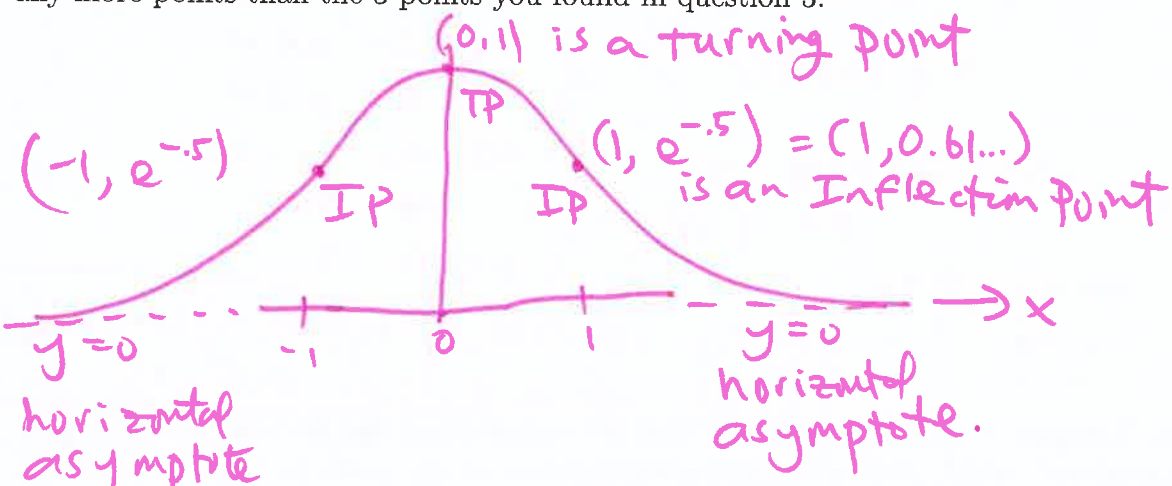
← Put them both together.

5. Evaluate these two limits using your intuition and knowledge of the graph $y = e^x$. $\lim_{x\to\infty} f(x) =$ 0 and $\lim_{x\to-\infty} f(x) =$ 0 . (No need to show work.)

for example, $f(\pm 10) = e^{-10^2/2} = e^{-50} = 1.9\ldots \times 10^{-22}$

Thus, the graph $y = f(x)$ has $y = 0$ as a horizontal asymptote.
(on both $x \to \infty$ and $x \to -\infty$)

6. Sketch the graph $y = f(x)$, using everything you have learned, but without plotting any more points than the 3 points you found in question 3.



Note that $f(-x) = e^{-(-x)^2/2} = e^{-x^2/2} = f(x)$.

Thus, f is even, and the graph is symmetric under a ~~refaltion~~ reflection across the y-axis.

See the desmos graph of this beautiful function!