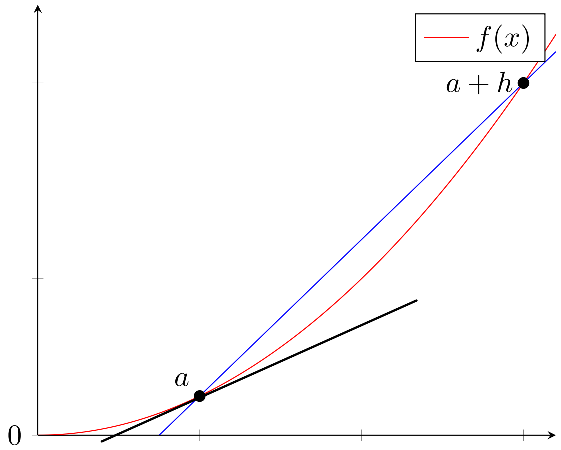
MAT 136 Week 4, Day 1: Limit Definition of the Derivative

The average rate of change of a function f from $x = a$ to $x = b$ is the slope of the secant line to the graph of $y = f(x)$ that passes through $(a, f(a))$ and $(b, f(b))$.

The **derivative** of the function f at the point $x = a$ is the instantaneous rate of change at $x = a$.

It is also the slope of the tangent to the graph of $f(x)$ at the point $(a, f(a))$. We looked at this scenario in Week 1, Day 4.



— secant line from $(a, f(a))$ to $(a+h, f(a+h))$

— tangent line at $(a, f(a))$.

To find the slope of the tangent at $(a, f(a))$, we can start with a secant through another point on the graph (say, $(a + h, f(a + h))$), and then let h approach 0. Now that we have limits, we can do this formally.

Definition. Let f be a function and let $a \in \mathbb{R}$. The **derivative** of f at $x = a$, if it exists, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if this limit exists).

Sometimes the slope of the secant line to $y = f(x)$ going through the points $(a, f(a))$ and $(a + h, f(a + h))$ is called a difference quotient.

- So the derivative of f at $x = a$, if it exists, is a limit of difference quotients.

Example 1. Consider the function $f(x) = -2x+5$. For each real number a, determine the derivative of f at the point a.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{-2(a+h)+5 - (-2a+5)}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{-2a}-2h+\cancel{5}+\cancel{2a}-\cancel{5}}{h}$$
$$= \lim_{h\to 0} \frac{-2h}{h} = \lim_{h\to 0} -2 = -2.$$

> If $f(x) = mx + b$ (where m, b are constants), then for each a the derivative of f at a is equal to m.
>
> • Note: m is the slope of this line!

Example 2. Find an equation for the tangent line to the graph $y = x^2$ at $x = 5$.

• First, find the derivative of $f(x)=x^2$ at $x=5$:

$$f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$
$$= \lim_{h\to 0} \frac{(5+h)^2 - 5^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{25}+10h+h^2-\cancel{25}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(10+h)}{\cancel{h}} = \lim_{h\to 0}(10+h) = 10.$$

So the slope of the tangent is 10.

• The tangent is at $(5, f(5)) = (5, 25)$.

Point slope form:

$$y - 25 = 10(x-5)$$
$$\Rightarrow\quad y-25 = 10x-50$$
$$\Rightarrow\quad y = 10x-25$$

both acceptable

slope, y-intercept form.

is an equation of the tangent line to $y=x^2$ at $x=5$.

Example 3. The controls on your remote control car are malfunctioning, causing your car to accelerate linearly. As a result, the current position $p(t)$ of your car from your starting point (in feet) after t seconds can be modelled by $p(t) = t^3 + 8t$. Determine the instantan~~d~~eous velocity of your car at $t = 2$ seconds.

· We want the derivative of $p(t)$ at $t=2$:

$$p'(2) = \lim_{h\to 0} \frac{p(2+h)-p(2)}{h}$$
$$= \lim_{h\to 0} \frac{(2+h)^3 + 8(2+h) - (2^3+8(2))}{h}$$
$$= \lim_{h\to 0} \frac{(2+h)(4+4h+h^2)+16+8h-(8+16)}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{8}+12h+6h^2+h^3+\cancel{16}+8h-\cancel{8}-\cancel{16}}{h}$$
$$= \lim_{h\to 0} \frac{20h+6h^2+h^3}{h}$$
$$= \lim_{h\to 0} (20+6h+h^2)$$
$$= 20.$$

Thus, the inst. velocity at $t=2$ seconds is 20 feet/sec.

Example 4. Find a formula for the instantaneous rate of change of the function $f(x) = \frac{1}{x}$ at the point $x = a$, in terms of a. (Assume that $a \neq 0$.)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

Common denominator:

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{a}{a(a+h)} - \frac{(a+h)}{a(a+h)}\right)$$
$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{a(a+h)}\right)$$
$$= \lim_{h\to 0}\left(-\frac{1}{a(a+h)}\right)$$
$$= -\frac{1}{a(a+0)} = -\frac{1}{a^2}.$$