MAT 137 (Calculus II) Prof. Swift

Worksheet on Radius of Convergence and Interval of Convergence of Power Series

Use the Ratio Test to determine the radius of convergence of each of these power series. Then find the interval of convergence by checking the endpoints, if necessary. (Note that the "interval" of convergence can be a single point.)

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$

3. $\sum_{n=0}^{\infty} n!x^n$

RATIO TEST

Recall: $\sum_{n=?}^{\infty} a_n$ converges absolutely if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\sum a_n$ diverges if $L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.

Test is inconclusive if $L = 1$.

1. $a_n = \frac{x^n}{n!}$, so $L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right|$

$= |x| \cdot \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$

thus, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is ∞.

2. $a_n = \frac{(2x)^n}{\sqrt{n}}$, so $L = \lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{\sqrt{n+1}}}{\frac{(2x)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right|$

$= |2x| \cdot \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = |2x| \cdot 1 = |2x|.$

The series converges absolutely if $|2x| < 1$, or $|x| < \frac{1}{2}$.

the series diverges if $|2x| > 1$, or $|x| > \frac{1}{2}$.

The radius of convergence is $R = \frac{1}{2}$. continued!

2, continued.

We need to check the 2 cases where $|2x|=1$, and the ratio test is inconclusive.

$x=\frac{1}{2}$ gives $\sum_{n=1}^{\infty} \frac{(2\cdot\frac{1}{2})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges, since it is a p-series with $p=\frac{1}{2}\leq 1$.

$x=-\frac{1}{2}$ gives $\sum_{n=1}^{\infty} \frac{(2(-\frac{1}{2}))^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the alternating series test.

$\boxed{\text{The interval of convergence is } [-\frac{1}{2}, \frac{1}{2}).}$

3. Every power series converges at its center, so $\sum_{n=0}^{\infty} n!\,x^n$ converges at $x=0$.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = \lim_{n\to\infty} |(n+1)\cdot x| = |x| \lim_{n\to\infty}(n+1) = \infty.$$

Therefore the P.S. diverges for all $x\neq 0$.

$\boxed{\begin{array}{l}\text{The radius of convergence is } R=0.\\ \text{The interval of convergence is } \{0\}.\end{array}}$