MAT 201 Problems, Fall 2011 updated December 1, 2011

Problems from Sept. 1

1. Solve $|x-2|+|x+2|=4$.

2. We have three apples of different sizes and five bananas of different sizes. We need to divide them into two packs of four fruits, each of which contains at least one apple. In how many different ways can we do that?

3. Analyze the game "Toot and Otto". Does the first player win if he or she uses perfect strategy? Is it always a draw if both players use perfect strategy? Those questions are probably too difficult, so *any* partial analysis of the game is welcome.

Outline of rules for Toot and Otto. Each person has 6 "T" tiles and 6 "O" tiles. The players drop them in a 6 wide by 4 high tray (like the board for connect 4). One person tries to spell TOOT across, down or diagonally. The other tries to spell OTTO.

4. List the integers greater than 10 and less than 100, written in base 10 notation, that are increased by 9 when their digits are reversed.

5. Triangle $\triangle A_1A_2A_3$ is equilateral. For all positive integers n, A_{n+3} is the midpoint of line segment A_nA_{n+1}. What is the measure of angle $\angle A_{44}A_{45}A_{43}$?

Problems for Sept. 8

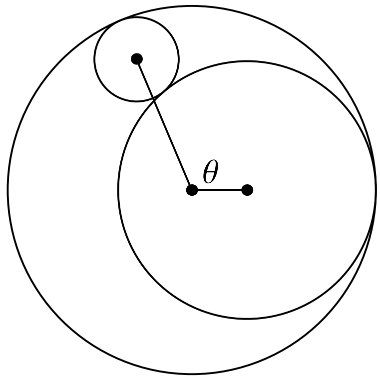
6. The following four statements, and only these, are found on a card.

On this card exactly one statement is false.
On this card exactly two statements are false.
On this card exactly three statements are false.
On this card exactly four statements are false.

Exactly how many of these statements are false?

7. Alice, Bob, and Caitlin are playing ping-pong. After every game, the loser sits down and the winner plays the one who sat out that game. After several hours, Alice has played 10 games, Bob has played 15, and Caitlin has played 17. Who lost the second game?

8. A, B, and C are the centers of three circles, each of which is tangent to each of the others. Show that the perimeter of the triangle ABC is equal to the diameter of the largest of the circles.



9. (Hint: Look up circles of Apollonius on the web.) I fudged that picture for problem 8. I started with a circle of radius 1, centered at the origin. That is the big circle. Then I put in a circle of radius 0.7 centered at $(0.3, 0)$. For the third, smallest, circle I found that a circle of radius 0.23 centered at $(-0.3, 0.712)$ looked good. But it is not perfectly tangent to the other 2 circles. Find the possible centers and radii (exactly) of the third circle. Or find one circle that works exactly. If possible, generalize so that the radius 0.7 of the second circle can be any number between 0 and 1.

Restatement of question 9: Let θ be the angle between the centers of the circles, as shown in the updated figure. Find r_3 (the radius of the third circle) as a function of θ and r_2. (For the figure, $r_2 = 0.7$). This effectively answers the original question since the center of the third circle is at $[1 - r_3](\cos\theta, \sin\theta)$

Problems from Sept. 15

10. (in-class quickie) If i is the number whose square is -1, what is $(1 + i)^2$?

11 (in-class quickie) What is i^{2011}?

12. Some students asked for a problem that requires differential equations. Here's one that I think can be solved with DEs, but frankly I don't know the solution.

A fox runs at exactly the same speed, v, as his prey, a rabbit. The fox starts at the origin, with coordinates $(0, 0)$. The rabbit starts one meter north of the fox, at $(0, 1)$. At $t = 0$, the rabbit starts running east, so his position is $x = vt, y = 1$. The fox also starts running at $t = 0$, and always runs straight at the rabbit, with speed v. The frustrated fox will never catch the rabbit.

12a. What curve in the $x-y$ plane will the fox follow? (It starts out with a vertical tangent, like $y = \sqrt{x}$, and has a horizontal asymptote.)

12b. As $t \to \infty$, how far from the rabbit will the fox be?

13. Let $u = \cot(22.5°)$. Prove that u satisfies a quadratic equation with leading coefficient 1 and integer coefficients.

14. The first 2011 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward, like 12021.

Problems from Sept. 22

15. Prove that any quadratic expression $Q(x) = Ax^2 + Bx + C$ can be put uniquely into the form $Q(x) = \frac{k}{2}x(x-1) + \ell x + m$, where k, ℓ, and m depend on the coefficients A, B, and C. Furthermore, prove that $Q(x)$ is an integer for all integers x if and only if k, ℓ, and m are integers.

16. Find the limit of this sequence as $n \to \infty$:

$$a_1 = \sqrt{2},\ a_2 = \sqrt{2\sqrt{2}},\ a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

17. (Requires Calculus III). Evaluate $\int_0^a \int_0^b e^{\max(b^2x^2,\, a^2y^2)}\, dy\, dx$, where a and b are positive.

Problems from Sept. 29

18. A large bucket has 2011 red balls and 2011 white balls, well mixed. Do this until there are fewer than three balls left:

- Pull out three balls at random.
- If the balls are all the same color, throw them all away.
- If there are two of one color and one of the other, throw away the single ball and stir the two of the same color back into the bucket.

What is the probability of each of the 6 possibilities for what is left when you are done?

- No balls
- One red ball
- One white ball
- Two red balls
- Two white balls
- One of each

Note added 10-15: You may use a computer to estimate the probabilities using a Monte Carlo method. But Steve Wilson assures me that this problem can be solved exactly with pencil and paper, at least with 2010 balls.

19. The escalator at the new Health and Learning Center (HLC) will take me one floor up in 15 seconds if I'm standing. If I walk up the moving escalator the time is reduced to 6 seconds. When the escalator is not moving, how long will it take me to walk up the escalator steps. (Ignore relativistic effects.)

20. Redo the escalator problem with the 15 replaced by t_1 and 6 replaced by t_2. Find the last time, t, in terms of t_1 and t_2. At this point the question becomes open ended: Find another choice for t_1 and t_2 that makes t an integer. Find more choices like this. Can you find all choices like this?

Problems from October 6.

21. (In class quickie) Without a calculator, determine the order of 10^8, 5^{12}, and 2^{24}. Your answer will be something like "$10^8 < 5^{12} < 2^{24}$", with justification.

22. How many whole numbers between 1 and 1000 do *not* contain the digit 1?

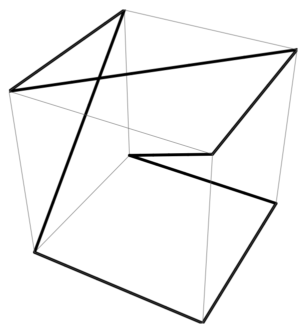
23. Find all of the right triangles which have integer length legs (that is, a and b are integers in the famous $a^2 + b^2 = c^2$) *and* the area of the triangle is numerically equal to three times the perimeter of the triangle.

Problems from October 13.

24. The polynomial $x^3 - ax^2 + bx - 210$ has three positive integer zeros. What is the smallest possible value of a?

25. An infinitesimal fly is inside a cubical box with side length 1 meter. It decides to visit each corner of the box. It will begin and end at the same corner, and visit

each of the other corners exactly once. It will either crawl or fly in a straight line from one corner to another. A picture of one possible path is shown. What is the maximum possible length of its path?



26. Eight distinct points are placed on a circle. Chords are drawn between every pair of points. The points have been placed so that no three chords intersect at a single point. How many triangles are drawn?

Problems from October 20.

27. Find all of the solutions to $x = |2x - |60 - 2x||$.

28. Find the smallest positive integer n such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square.

29. A geometric sequence $\{a_n\}_{n=1}^{\infty}$ has $a_1 = \sin(x)$, $a_2 = \cos(x)$, and $a_3 = \tan(x)$ for some real number x. For what value of n does $a_n = 1 + \cos(x)$?

Problems from October 27.

30. (Modification of problem 18) The setup is like problem 18. However, the second rule is modified:

- Pull out three balls at random.
- (1) If the balls are all the same color, throw them all away.
- (2) If there are two of one color and one of the other, stir one of the majority color back into the bin, and throw away the other two.

(The old rule 2 is this: If there are two of one color and one of the other, throw away the single ball and stir the two of the same color back into the bucket.)

One more thing. Steve Wilson's original problem had 2010 balls, and that might make a difference. Try it with both cases: 2010 balls or 2011. Does it matter?

31. (Modification of problem 26.) Eight distinct points are placed on a circle. Chords are drawn between every pair of points. The points have been placed so that no three chords intersect at a single point. How many triangles are drawn **with all three vertices inside the circle**?

32. Is problem 26 well posed? In other words, do you always get the same number of triangles for any placement of the 8 vertices?

Problems from November 3.

33. If you write down every integer from 1 to 1,000,000, how many times do you write the digit 1?

34. Three people make these statements:

"My name is Aaron, I'm an architect, and I live in Austin."

"My name is Bob, I'm a baker, and I live in Birmingham."

"My name is Charlie, I'm a carpenter, and I live in Chicago."

Well, the three speakers are indeed Aaron, Bob and Charlie. Their jobs are architect, baker and carpenter, and they do live in Austin, Birmingham, and Chicago.

The problem is that they are all liars. Each person told three lies! If Aaron does live in Austin, but is a baker, who is the first speaker, what does he do, and where is he from?

35. A box contains red and yellow balls. If one red ball were removed, one-seventh of the remaining balls would be red. If 5 yellow balls were removed instead, one-sixth of the remaining ones would be red. How many balls are there in the box?

Problems from November 10.

36. Sally is making a scale model of her town. The town's water tower is 50 meters high, and the top portion is a sphere that holds 10^6 liters of water. The sphere in Sally's miniature water tower holds 1 milliliter of water. How tall is the tower?

37. In a remote village there are two kinds of people: liars and truth-tellers. The liars always lie and the truth-tellers always tell the truth. Brian, Chris, LeRoy, and Mike live together in this village, and they make the following statements.

Brian: " Mike and I are different kinds of people."

Chris: “ LeRoy is a liar.”

LeRoy: “Chris is a liar.”

Mike: “Of the four of us, at least two are truth-tellers.”

How many of these four people are liars?

Problems from November 17

38. The fifth and eighth terms of a geometric sequence of real numbers are 7! and 8! respectively. What is the first term?

39. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 sets, each sitting in the seat she was before or a seat next to the one she occupied before. In how many ways can the women be reseated?

40. For what values of x in $[0, \pi]$ is $\arcsin(\sin(6x)) = \arccos(\cos(x))$? Do not use a calculator, except possibly to check your answer.

Problems from December 1

41. The length of the interval of solutions of the inequality $a \leq 2x + 3 \leq b$ is 10. What is $b - a$?

42. A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume of the remaining solid?

43. Alice drove at an average rate of 80 kph and then stopped for 20 minutes for gas and a snack. After the stop, she drove at an average rate of 100 kph. Altogether she drove 250 km in a total trip time of 3 hours, including the stop. How many hours did she drive before the stop?

updated December 1, 2011