MAT 216, Introduction to Matrix Algebra. Solution by Jim Swift at NAU.

The augmented matrix for my version of problem 8 in set 4 is

$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 2 \end{bmatrix}$. The coefficient matrix is the same for everybody. Consider

$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 1 & 0 & 0 & 1 & d \end{bmatrix}$. Do the row operation $R_4 - R_1 \to R_4$ to get

$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & -1 & 0 & 1 & d-a \end{bmatrix}$. Do the row operation $R_4 + R_2 \to R_4$ to get

$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 1 & 1 & d-a+b \end{bmatrix}$. Do the row operation $R_4 - R_3 \to R_4$ to get

$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 0 & d-a+b-c \end{bmatrix}$. There are no solutions if $d - a + b - c \neq 0$, i.e. $d \neq a - b + c$.

Assume $d = a - b + c$, which is the case for all versions of this problem. The matrix becomes

$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Do the row operation $R_1 - R_2 \to R_1$ to get

$\begin{bmatrix} 1 & 0 & -1 & 0 & a-b \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Do the row operation $R_1 + R_3 \to R_1$ to get

$\begin{bmatrix} 1 & 0 & 0 & 1 & a-b+c \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Do $R_2 - R_3 \to R_2$ to get

$\begin{bmatrix} 1 & 0 & 0 & 1 & a-b+c \\ 0 & 1 & 0 & -1 & b-c \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore x_4 is a free variable, and the solution is

$(x_1, x_2, x_3, x_4) = (a-b+c-x_4, b-c+x_4, c-x_4, x_4) = (a-b+c, b-c, c, 0) + x_4(-1, 1, -1, 1)$.

My version of the problem has $a = -1$, $b = -4$, $c = -1$, and $d = 2$. Note that $a - b + c = -1 + 4 - 1 = 2 = d$, so the system is consistent. There are infinitely many solutions to my version of the problem, namely

$(x_1, x_2, x_3, x_4) = (-1+4-1-x_4, -4+1+x_4, -1-x_4, x_4) = (2, -3, -1, 0) + x_4(-1, 1, -1, 1)$.