Triple Integrals in Spherical Coordinates

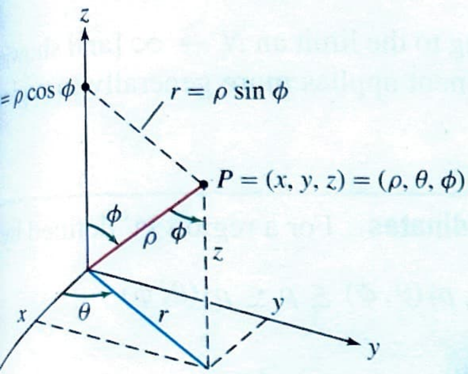


FIGURE 13 Spherical coordinates.

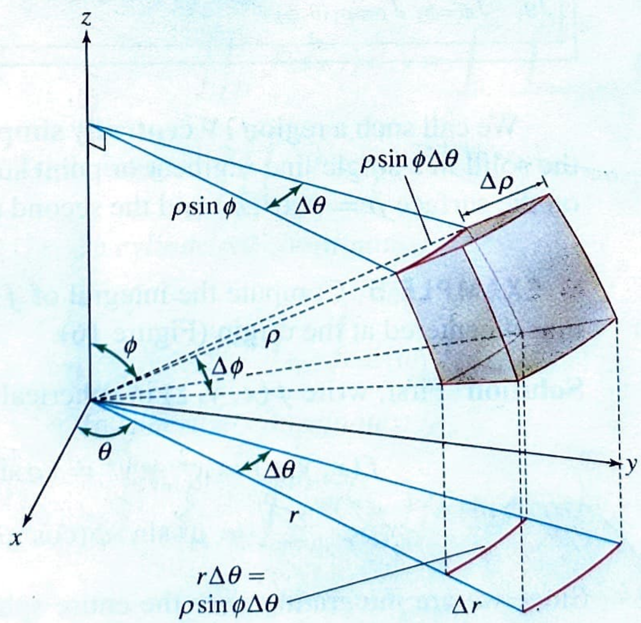
We noted that the Change of Variables Formula in cylindrical coordinates is summarized by the symbolic equation $dV = r\,dr\,d\theta\,dz$. In spherical coordinates (ρ, θ, ϕ) (introduced in Section 12.7), the analog is the formula

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Recall (Figure 13) that

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi, \qquad r = \rho \sin\phi$$

The key step in deriving this formula is estimating the volume of a small **spherical wedge** $\mathcal{W}$. Suppose it is defined by fixing values for ρ, ϕ, and θ, and varying each coordinate by a small amount given by $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$ as in Figure 14.



For small increments, the wedge is nearly a rectangular box with dimensions $\rho\sin\phi\Delta\theta \times \rho\Delta\phi \times \Delta\rho$.

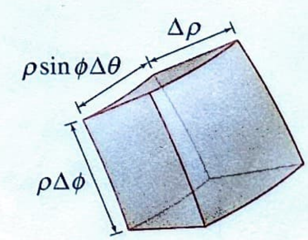


FIGURE 14 Spherical wedge.

The spherical wedge is nearly a box with sides $\Delta\rho$, $\rho\Delta\phi$, and $r\Delta\theta$ by projecting up from the corresponding length in the xy-plane. Converting $r\Delta\theta$ to spherical coordinates, we obtain $\rho \sin\phi\,\Delta\theta$ for this third dimension of the box.

Therefore, the volume of the spherical wedge is approximately given by the product of these three dimensions, the accuracy of which improves the smaller we take our changes in the variables:

$$\text{Volume}(\mathcal{W}) \approx \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta \qquad 6$$