

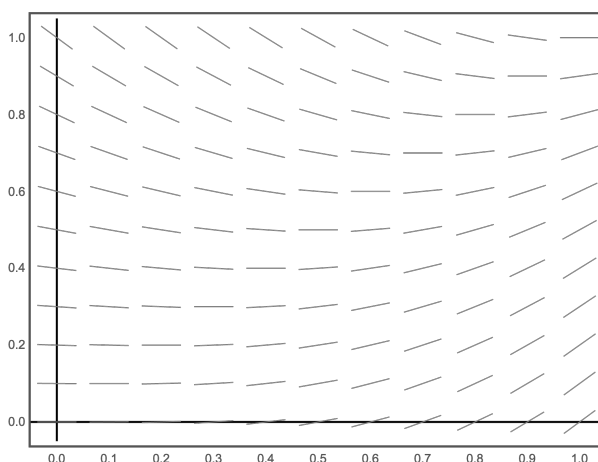
## Euler's Method with a Spreadsheet

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**Euler's method** is a way to find approximate solutions to an Initial Value Problem (IVP). You won't find the general solution since you need an Initial Condition (IC). The slope field applet written by Darryl Nester (linked to on our web site) uses Euler's method or other more accurate methods to compute approximate solutions to an IVP. This is useful for drawing the pictures, but also because often we cannot find a closed form solution. For example, the IVP

$$\frac{dy}{dx} = x^2 - y^2, \quad y(0) = 1, \quad 0 \leq x \leq 1$$

does not have a solution that can be written in terms of sines and cosines, exponential and logs, powers and roots, etc.



Euler's method is intimately related to the slope field. To start out, put a dot at the initial condition  $(x_0, y_0) = (0, 1)$ . Sketch the solution to the ODE that goes through that point. At every point in the graph of the solution, the slope of the solution curve is the slope of the line segment that would be there if there were a line segment at every point.

Now we will do Euler's method in three ways: first graphically, then numerically with a calculator, then we automate the calculations with a spreadsheet on your phone or laptop. Euler's method always needs a *step size*, which is called  $h$ , but could be called  $\Delta x$ . We will start with  $h = 0.5$  to get approximate values of  $y(0.5)$ , and  $y(1)$ .

Starting at the initial point  $(x_0, y_0) = (0, 1)$ , use an ID card to draw a line segment with the indicated slope between  $x = 0$  and  $x = 0.5$ . The slope is  $0^2 - 1^2 = -1$ , so  $\Delta y = h * (-1) = -0.5$ . Put a dot at the right endpoint, which is  $(x_1, y_1) = (0.5, 0.5)$ . The slope at that new point is  $.5^2 - .5^2 = 0$ . Use the slope at that dot to draw a horizontal line segment between  $(0.5, 0.5)$  and  $(x_2, y_2) = (1, 0.5)$ .

That is a poor approximation of the solution! We estimate that  $y(1) = 0.5$  using Euler's method with  $h = 0.5$ . To get a better approximation, do Euler's method graphically again, this time with step size  $h = 0.25$ . The starting point is  $(x_0, y_0) = (0, 1)$  as before. This time,  $\Delta y = h * (-1) = -0.25$ . The first point is  $(x_1, y_1) = (0.25, 0.75)$ . Since there is no slope line at this new point, you have to interpolate the nearby slopes to get to  $(x_2, y_2)$ . Continue in this way until you get to  $x_4 = 1$ .

**The formulas for Euler's method** to approximate the solution to the IVP  $y' = f(x, y)$ ,  $y(x_0) = y_0$  with step size  $h$  are

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + h * f(x_n, y_n), \quad \text{for } n = 0, 1, 2, \dots .$$

Use these formulas to fill in the table, with the help of a calculator. Use step size  $h = 0.25$  to reproduce the second approximation you found graphically on the other side of this sheet.

$$\frac{dy}{dx} = x^2 - y^2, \quad y(0) = 1, \quad 0 \leq x \leq 1$$

$n$	$x_n$	$y_n$
0	0	1
1		
2		
3		
4		

Here's how to fill in the table: Start with  $x_0 = 0$  and  $y_0 = 1$  since the IC is  $y(0) = 1$ . We know from the start that  $x_1 = h = .25$ ,  $x_2 = .5$ ,  $x_3 = .75$  and  $x_4 = 1$ . Now compute  $y_1$  using the formula with  $f(x, y) = x^2 - y^2$ , and fill in the blank and put it in the table above:

$$y_1 = y_0 + h * (x_0^2 - y_0^2) = 1 + .25 * (0^2 - 1^2) = \underline{\hspace{2cm}}$$

Now that you know  $y_1$  you can compute  $y_2$  with the formula

$$y_2 = y_1 + h * (x_1^2 - y_1^2) = y_1 + .25 * (.25^2 - y_1^2) = \underline{\hspace{2cm}}$$

Continue in this way to fill in the blanks:

$$y_3 = y_2 + h * (x_2^2 - y_2^2) = y_2 + .25 * (.5^2 - y_2^2) = \underline{\hspace{2cm}}$$

$$y_4 = y_3 + h * (x_3^2 - y_3^2) = y_3 + .25 * (.75^2 - y_3^2) = \underline{\hspace{2cm}}$$

You should find that  $y_4 \approx 0.6435$ . This means that you have found that  $y(1) \approx 0.6435$  on an approximate solution using Euler's method with a step size of  $h = 0.25$ .

**Euler's method** is easy to implement **on a spreadsheet** like Excel or Google Sheets. You don't need to keep track of the  $n$  like the previous calculation. Input these formulas and then *fill down* as far as you need.

	$A$	$B$
1	$x$	$y$
2	0	1
3	$= A2 + .25$	$= B2 + .25 * (A2^2 - B2^2)$

You will want this for WeBWorK, where some problems ask for 20 steps of Euler's method.

Now try it again with a step size of  $h = 0.125$ . You should find  $y(1) \approx 0.7003$  with a step size of 0.125.

Finally, approximate  $y(1)$  for this IVP with a step size of  $h = 0.05$ . You should find that  $y(1) \approx 0.7309$ . This is about 2.6% off from the true value of  $y(1) = 0.75002\dots$ . Even with this small step size, Euler's method is not very accurate. That is why the slope field applet has other methods besides Euler's method that it can use.