Some ODEs we can Solve by Inspection

by Jim Swift @ NAU

This page gives the solutions to a few families of ODEs that comprise the majority of ODEs that we need to solve in practice. Learn to solve these by inspection.

(1) The IVP for exponential growth ($k > 0$) or decay ($k < 0$):

$$\frac{dy}{dt} = ky, \; y(0) = y_0 \text{ has the solution } y = y_0 e^{kt}.$$

(2) The IVP for the motion of a particle with no external force (acceleration is 0):

$$\frac{d^2y}{dt^2} = 0, \; y(0) = y_0, \; y'(0) = v_0 \text{ has the solution } y = y_0 + v_0 t.$$

(3) We can generalize example (2) without much work:

$$\frac{d^m y}{dt^m} = 0 \text{ has the general solution } y = c_1 + c_2 t + c_3 t^2 + \cdots c_m t^{m-1}.$$

(4) The IVP for Simple Harmonic Motion in physics (a mass on a spring with no friction) is super important:

$$\frac{d^2y}{dt^2} = -\omega_0^2 y, \; y(0) = y_0, \; y'(0) = v_0 \text{ has the solution } y = y_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t).$$

In math classes we often use "k" instead of "ω_0". Note that y'' equals a *negative* constant times y, and the characteristic equation is $r^2 = -\omega_0^2$, which has the roots $r = \pm i\omega_0$.

(5) Finally, there is another family of ODEs that comes up frequently in the study of Partial Differential Equations; here y'' equals a *postive* constant times y:

$$\frac{d^2y}{dt^2} = k^2 y$$

The characteristic equation is $r^2 = k^2$, so the roots are $r = \pm k$. The general solution to $y'' = k^2 y$ can be written as

$$y = c_1 e^{kt} + c_2 e^{-kt}.$$

But there is another form of the general solution that is useful for solving initial value problems: The general solution to $y'' = k^2 y$ can be written as

$$y = c_1 \cosh(kt) + c_2 \sinh(kt),$$

where $\cosh(t) := (e^t + e^{-t})/2$ and $\sinh(t) := (e^t - e^{-t})/2$. These have the properties $\frac{d}{dt}\cosh(t) = \sinh(t)$ and $\frac{d}{dt}\sinh(t) = \cosh(t)$, with no minus signs to remember unlike cosine and sine. Using this second form of the general solution, we find that

$$\frac{d^2y}{dt^2} = k^2 y, \; y(0) = y_0, \; y'(0) = v_0 \text{ has the solution } y = y_0 \cosh(kt) + \frac{v_0}{k} \sinh(kt).$$