MAT 239 (Differential Equations), Prof. Swift
Worksheet 29, Eigenvalue hack, and the Swift method.

The eigenvalues λ_1 and λ_2 of a matrix 2×2 matrix A, satisfy

$$\lambda_1 + \lambda_2 = T \text{ and } \lambda_1 \cdot \lambda_2 = D$$

where $T = \mathrm{Tr}(A)$ is the trace of A (the sum of the diagonal entries) and $D = \mathrm{Det}(A)$ is the determinant of A. That is enough to find the eigenvalues of A in many cases. If that doesn't work, use the fact that the characteristic equation of A is $\lambda^2 - T\lambda + D = 0$.

1. Use these hacks to find the eigenvalues of the following matrices:

$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ $T = 3+4 = 7$, $D = 3\cdot 4 - 1\cdot 2 = 12 - 2 = 10$

$\lambda_1 + \lambda_2 = 7$, $\lambda_1 \cdot \lambda_2 = 10$

so $\boxed{\lambda = 2, 5}$

$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$ $T = 0 - 2 = -2$, $D = 0(-2) - 1(-1) = 1$

so $\lambda_1 + \lambda_2 = -2$, $\lambda_1 \cdot \lambda_2 = 1$

$\boxed{\lambda = -1, -1}$

$\begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$ $T = -3 + 1 = -2$, $D = -3 - (-8)$, $D = 5$

$\lambda_1 + \lambda_2 = -2$, $\lambda_1 \cdot \lambda_2 = 5$ can't be solved with integers.

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda^2 + 2\lambda = -5$

$\lambda^2 + 2\lambda + 1 = -5 + 1$

$(\lambda + 1)^2 = -4$

$\lambda + 1 = \pm 2i$

$\boxed{\lambda = -1 \pm 2i}$

2. Fill in the first row so the eigenvalues are 3 and -2.

$\begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 2 \end{bmatrix}$ $T = 3 - 2 = 1$, $D = 3(-2) = -6$

$T = a + 2 = 1$ $\therefore \underline{a = -1}$

$D = 2a - b = -6$ $\therefore 2(-1) - b = -6$

$-b = -6 + 2 = -4$

$\underline{b = 4}$

3. Use the Swift method to solve the IVP $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$.

Note: Paul does this the hard way in example 3 of his notes on complex eigenvalues.

$T = 3+1 = 4,\ D = 3 - (5)(-13) = 3+65 = 68 \quad \lambda_1 + \lambda_2 = 4,\ \lambda_1 \cdot \lambda_2 = 68.??$

Use char. eqn. $\lambda^2 - 4\lambda + 68 = 0$

$$\lambda^2 - 4\lambda = -68$$
$$\lambda^2 - 4\lambda + 4 = -68 + 4$$
$$(\lambda - 2)^2 = -64$$
$$\lambda - 2 = \pm 8i, \quad \boxed{\lambda = 2 \pm 8i}$$

$a=2,\ b=8$ in formulas.

$\vec{x}_0 = \begin{bmatrix} 3 \\ -10 \end{bmatrix},\ \vec{x}_1 = \begin{bmatrix} 3-2 & -13 \\ 5 & 1-2 \end{bmatrix}\vec{x}_0 = \begin{bmatrix} 1 & -13 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -10 \end{bmatrix}$

$\vec{x}_1 = \begin{bmatrix} 3+130 \\ 15+10 \end{bmatrix} = \begin{bmatrix} 133 \\ 25 \end{bmatrix}$. So the formula says

$$\boxed{\vec{x}(t) = e^{2t}\left(\begin{bmatrix} 3 \\ -10 \end{bmatrix}\cos(8t) + \begin{bmatrix} 133 \\ 25 \end{bmatrix}\frac{\sin(8t)}{8}\right)}$$