MAT 239, Differential Equations, Prof. Swift

The solution to any IVP for 2 linear, homogeneous 1st order ODEs with constant coefficients

Systems of ODEs

These formula concern the solution to the Initial Value Problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where A is a 2×2 matrix with constant, real entries, and $\mathbf{x}_0$ is a 2-dimensional vector with real components. Note that $\mathbf{x}' = \frac{d\mathbf{x}}{dt}$ is sometimes used, and other variables besides $\mathbf{x}$ are sometimes used.

The eigenvalues of A satisfy $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$, and the associated eigenvectors satisfy $A\mathbf{v} = \lambda\mathbf{v}$, or $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, is a solution to the ODE $\mathbf{x}' = A\mathbf{x}$, for real or complex eigenvalues λ. But complex eigenvalues have complex eigenvectors and $e^{\lambda t}\mathbf{v}$ is a complex-valued vector. With repeated eigenvalues, we cannot find two linearly independent eigenvectors.

Case 1. A has real, distinct eigenvalues, $\lambda_1 \neq \lambda_2$.
The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. The constants c_1 and c_2 are determined by the initial condition.

Case 2. A has complex eigenvalues, $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, with $b > 0$.
Compute $\mathbf{x}_1 = (A - aI)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 \cos(bt) + \mathbf{x}_1 \tfrac{1}{b}\sin(bt)\right)$$

Case 3. A has repeated, real eigenvalues, $\lambda_1 = \lambda_2$.
Compute $\mathbf{x}_1 = (A - \lambda_1 I)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{\lambda_1 t}\left(\mathbf{x}_0 + \mathbf{x}_1 t\right)$$

Note that cases 2 and 3 are actually easier computations. I don't think these formulas are in Paul's notes or our suggested textbook. If you see them somewhere on the web, send me the link!

In case 2, an eigenvector for λ_1 is $\mathbf{x}_0 - i\frac{1}{b}\mathbf{x}_1$, and this formula follows from the general solution given in the book and Paul's notes.

In case 3, if $\mathbf{x}_0$ is an eigenvector, then $\mathbf{x}_1 = \mathbf{0}$ and the solution is simply $\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{x}_0$. If $\mathbf{x}_0$ is not an eigenvector, then $\mathbf{x}_1 \neq \mathbf{0}$ is an eigenvector. The solution given follows from Paul's notes.

Note that case 3 is obtained from case 2 in the limit $b \to 0$, since $\lim_{b\to 0}\cos(bt) = 1$ and $\lim_{b\to 0}\frac{1}{b}\sin(bt) = t$.