MAT 239 (Differential Equations), Prof. Swift
Worksheet 14, The Characteristic Equation

This whole worksheet is about the ODE $y'' + y' - 2y = 0$. Assume that the independent variable is x, so $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$.

1. Write down one solution of the ODE. $y = 0$

2. Plug the function $y = e^{rx}$ into the ODE and find the equation that the constant r must satisfy so that $y = e^{rx}$ is a solution to the ODE. This is called the *characteristic equation* of the ODE, and it is super important.

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

$y'' + y' - 2y = 0$ becomes

$r^2e^{rx} + re^{rx} - 2e^{rx} = 0$

$e^{rx}(r^2 + r - 2) = 0$ for all x.

e^{rx} is Never 0. so $\boxed{r^2 + r - 2 = 0}$

We found that the characteristic equation of $y'' + y' - 2y = 0$ is $r^2 + r - 2 = 0$

3. Find the two roots of the characteristic equation. Call them r_1 and r_2.

$(r-1)(r+2) = 0$, so $\boxed{r_1 = 1, r_2 = -2}$, or $r = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{9}}{2}$

$= \frac{-1 \pm 3}{2} = \frac{2}{2}, \frac{-4}{2}$

$= 1, -2$

4. Write down two different non-zero solutions of the ODE. Call them y_1 and y_2.

$y_1 = e^{1x} = e^x$, $y_2 = e^{-2x}$

5. The general solution to the ODE is $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are arbitrary constants. Write down the general solution using the solutions you found in part 4. Does this include the solution you guessed in question 1?

$y = c_1 e^x + c_2 e^{-2x}$ yes: $\boxed{c_1 = c_2 \text{ gives } y = 0}$

6. Verify that the general solution you wrote down is a solution to the ODE for all c_1 and all c_2. (Evaluate y' and y'', plug these into the ODE, and gather terms.)

$y = c_1 e^x + c_2 e^{-2x}$

$y' = c_1 e^x - 2c_2 e^{-2x}$

$y'' = c_1 e^x + 4c_2 e^{-2x}$

$y'' + y' - 2y \stackrel{?}{=} 0$

$(c_1 e^x + 4c_2 e^{-2x}) + (c_1 e^x - 2c_2 e^{-2x}) - 2(c_1 e^x + c_2 e^{-2x}) \stackrel{?}{=} 0$

$(c_1 + c_1 - 2c_1) e^x + (4c_2 - 2c_2 - 2c_2) e^{-2x} \stackrel{?}{=} 0$

$0 e^x + 0 e^{-2x} \stackrel{\checkmark}{=} 0$