MAT 239 (Differential Equations), Prof. Swift
Worksheet 15, on finding the constants in *the* general solution

One version of WeBWorK problem 2 says "*The* general solution of a certain differential equation *can be written* as $y(t) = c_1 e^{\cancel{2y}\, 2t} + c_2 e^{5t}$." Then they ask you to solve an IVP. $y = C_1 e^{2t} + C_2 e^{5t}$

I have a different question for you, that will be super-important in this class, on the front page of this worksheet. On the back page you get some practice solving for c_1 and c_2, and also learn the reason for the italics in the quote of WeBWorK problem 2.

1. What is that certain ODE from the WeBWorK problem?

e^{2t} gives $r_1 = 2$, so $(r-2)(r-5) = 0$ is the characteristic eqn.
e^{5t} gives $r_2 = 5$ $\quad r^2 - 7r + 10 = 0$ " " " "

$\boxed{y'' - 7y' + 10y = 0 \text{ is the ODE}}$

2. Find the general solution to $y'' - y = 0$. Assume that t is the independent variable. You will use this general solution to solve 2 different IVPs on the back of this page.

Plug in $y = e^{rt}$ $\quad$ so $r^2 - 1 = 0$ is the characteristic equation
$r^2 e^{rt} - e^{rt} = 0$ $\quad$ or $r^2 = 1$ $\quad$ so $\boxed{y = C_1 e^{t} + C_2 e^{-t} \text{ is the general solution.}}$
$(r^2 - 1)e^{rt} = 0$ $\quad$ $\underline{r = \pm 1}$

I chose $r_1 = 1$, $r_2 = -1$.

3. Solve the IVP $y'' - y = 0, \quad y(0) = 1,\ y'(0) = 0$. Call this solution $y_a(t)$.

$y = C_1 e^t + C_2 e^{-t}$; $y(0) = C_1 e^0 + C_2 e^{-0} = \boxed{C_1 + C_2 = 1}$

$y' = C_1 e^t - C_2 e^{-t}$ $\quad y'(0) = C_1 e^0 - C_2 e^{-0} = \boxed{C_1 - C_2 = 0}$

add: $2C_1 = 1 \therefore C_1 = \frac{1}{2}$

Subtract: $2C_2 = 1 \therefore C_2 = \frac{1}{2}$

$\boxed{y_a(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}}$

4. Solve the IVP $y'' - y = 0, \quad y(0) = 0,\ y'(0) = 1$. Call this solution $y_b(t)$.

$C_1 + C_2 = 0$ $\quad$ add: $2C_1 = 1 \therefore C_1 = \frac{1}{2}$

$C_1 - C_2 = 1$ $\quad$ Subtract: $2C_2 = -1 \therefore C_2 = -\frac{1}{2}$

$\boxed{y_b(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}}$

5. The general solution to $y'' - y = 0$ can be written as $y(t) = ay_a(t) + by_b(t)$, where a and b are arbitrary constants. Find the constants c_1 and c_2 in terms of a and b, and then find a and b in terms of c_1 and c_2. Moral: *The* general solution *can be written* in many different ways.

$ay_a + by_b = a\left(\frac{1}{2}e^t + \frac{1}{2}e^{-t}\right) + b\left(\frac{1}{2}e^t - \frac{1}{2}e^{-t}\right) = \left(\frac{a+b}{2}\right)e^t + \left(\frac{a-b}{2}\right)e^{-t}$

$\therefore \boxed{C_1 = \frac{a+b}{2}}\ \boxed{C_2 = \frac{a-b}{2}}$

After getting C_1 and C_2; solve for a and b: $\boxed{\begin{aligned} a &= C_1 + C_2 \\ b &= C_1 - C_2 \end{aligned}}$

6. Use the *new* form of the general solution to solve the IVP, $y'' - y = 0, \quad y(0) = 7,\ y'(0) = -3$.

Use what you know about y_a and y_b to find a: $y(0) = ay_a(0) + by_b(0) = a \cdot 1 + b \cdot 0 = 7$ $\quad \therefore \boxed{a = 7}$

Use what you know about y_a and y_b to find b: $y'(0) = ay_a'(0) + by_b'(0) = a \cdot 0 + b \cdot 1 = -3$ $\quad \therefore \boxed{b = -3}$

The solution, written in terms of $y_a(t)$ and $y_b(t)$, is $\boxed{y = 7y_a(t) - 3y_b(t)}$