MAT 239 (Differential Equations), Prof. Swift
Worksheet 23, Driven Oscillators

The nondimensional form of the ODE for a driven, damped oscillator is

$$y'' + \frac{1}{Q}y' + y = \cos(\omega t),$$

where Q and ω are positive, dimensionless constants. The quality factor Q measures the friction (also called damping), and ω is the ratio of the driving frequency to the natural frequency ω_0.

1. Use the method of undetermined coefficients to find a particular solution to $y'' + y = \cos(\omega t)$ with $\omega \neq 1$. This is the limit of $Q \to \infty$, which means zero friction. Write down the form of the particular solution y_p, with constants A and B, then find A and B as functions of ω.

$y_p = A\cos(\omega t) + B\sin(\omega t)$. Rule 2 does not apply since $\omega \neq 1$.

$y_p' = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$

$y_p'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$

$y_p'' + y_p = A(1-\omega^2)\cos(\omega t) + B(1-\omega^2)\sin(\omega t) = 1\cos(\omega t) + 0\sin(\omega t)$

$A(1-\omega^2) = 1$, $B(1-\omega^2) = 0$

$\therefore A = \frac{1}{1-\omega^2}$, $B = 0$, and $\boxed{y_p = \frac{1}{1-\omega^2}\cos(\omega t)}$

2. Use the method of undetermined coefficients to find a particular solution to $y'' + \frac{1}{Q}y' + y = \cos(t)$. This is the case where $\omega = 1$. Write down the form of y_p, then find A and B as functions of Q.

$y_p = A\cos(t) + B\sin(t)$, Rule 2 does not apply, since Q is finite

$y_p' = -A\sin(t) + B\cos(t)$

$y_p'' = -A\cos(t) - B\sin(t)$

$y_p'' + \frac{1}{Q}y_p' + y_p = \cos(t)$ $\cancel{\cos\sin(t)}$ becomes

$$\underbrace{-A\cos(t) - B\sin(t)}_{\text{cancels!}} + \frac{1}{Q}\left(-A\sin(t) + B\cos(t)\right) + \underline{A\cos(t) + B\sin(t)} = \cos(t)$$

$$-\frac{A}{Q}\sin(t) + \frac{B}{Q}\cos(t) = \cos(t) + 0\sin(t)$$

$$-\frac{A}{Q} = 0, \text{ and } \frac{B}{Q} = 1 \quad \therefore A = 0,\ B = Q$$

Thus, $\boxed{y_p = Q\sin(t)}$