MAT 239 (Differential Equations), Prof. Swift
Worksheet 28, Eigenvalues and Eigenvectors

The eigenvalues/eigenvectors of a matrix A satisfy $A\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq \mathbf{0}$, and $\det(A - \lambda I) = 0$.

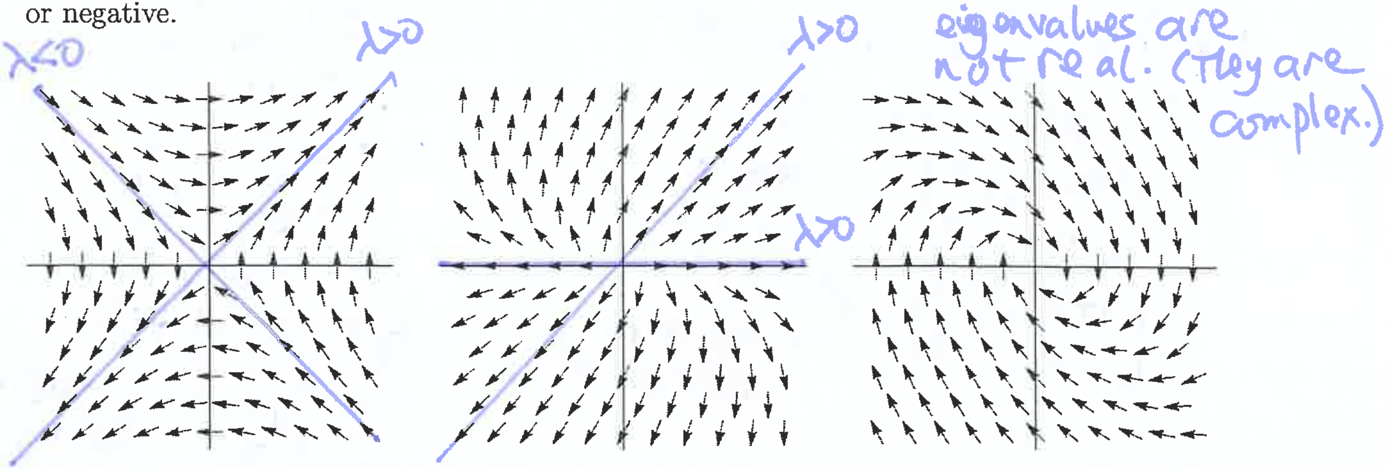
1. Let $A = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 5 & 1 \\ 0 & 1 & 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Yes/No: Is $A\mathbf{v} = 3\mathbf{v}$? $(\vec{0} = \vec{0})$

Yes/No: Does that imply that 3 is an eigenvalue of A? Why not? Because $\vec{v} = \vec{0}$ is not allowed

Yes/No: Is 3 an eigenvalue of A? (Hint: The determinant of $\begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 6 \end{bmatrix}$ is -6.) Why not? $\det(A-3I) \neq$

2. The figure shows 3 vector fields $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$. If A has real eigenvalues, draw the lines that are the eigenvector directions, and indicate if the corresponding eigenvalue is positive or negative.



2. Compute the eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix}1 & 1\\ 0 & 2\end{smallmatrix}\right]$.

$\det(A - \lambda I) = 0$

$\det\begin{bmatrix}1-\lambda & 1\\ 0 & 2-\lambda\end{bmatrix} = 0$

$(1-\lambda)(2-\lambda) = 0$

$\boxed{\lambda_1 = 1,\ \lambda_2 = 2 \text{ are the eigenvalues}}$

Let $\vec{v}_1 = \begin{bmatrix}a\\ b\end{bmatrix}$ be eigenvector with eigenvalue $\lambda_1 = 1$

$\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$

$\begin{bmatrix}a+b\\ 2b\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$

$a+b = a$
$2b = b$

$\therefore b = 0$

a = anything but 0.

$\boxed{\vec{v}_1 = \begin{bmatrix}1\\ 0\end{bmatrix} \text{ is one choice.}}$

Let's use the other form of $A\vec{v} = \lambda I$

$(A - \lambda I)\vec{v} = \vec{0}$.

Let $\vec{v}_2 = \begin{bmatrix}a\\ b\end{bmatrix}$ be the eigenvector with eigenvalue $\lambda_2 = 2$.

$(A - 2I)\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\begin{bmatrix}-a+b\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$-a+b=0$ so $a = b \neq 0$.

$\boxed{\vec{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}}$ is a natural choice.