Theory of Linear Homogeneous ODEs

by Jim Swift @ NAU

Consider the second order linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0, \text{ which can be written as } L[y] = 0.$$

Suppose that p and q are continuous on an open interval I, possibly $I = (-\infty, \infty)$, and $t_0 \in I$, and $y_1(t)$ and $y_2(t)$ are solutions to $L[y] = 0$ on I. Then the following are equivalent. (That means that either they are **all true**, or they are **all false**.)

- The general solution to $L[y] = 0$ on I is $y = c_1\, y_1(t) + c_2\, y_2(t)$.
- y_1 and y_2 are *linearly independent* functions on I. That is, neither of the functions is a constant multiple of the other function.
- $\Big(c_1\, y_1(t) + c_2\, y_2(t) = 0 \text{ for all } t \in I\Big) \Rightarrow \Big(c_1 = c_2 = 0\Big)$
  This is the usual definition of "linearly independent".
- $\{y_1, y_2\}$ is a fundamental solution set of $L[y] = 0$. (This is just a definition: a fundamental solution set of a second order linear homogeneous ODE is any set of two linearly independent solutions to the ODE.)
- The family of functions $y = c_1\, y_1(t) + c_2\, y_2(t)$ can be used to solve every IVP $L[y] = 0$, $y(t_0) = y_0$, $y'(t_0) = v_0$ (where y_0, v_0 are any constants).
- The Wronskian evaluated at t_0 is nonzero. That is, $W(y_1, y_2)(t_0) \neq 0$.
- The Wronskian is never 0 on I. That is, $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

The general solution of an nth order Linear Homogeneous ODE is a linear combination of a set of n linearly independent solutions. ("Linearly independent" means that none of the functions y_i can be written as a linear combination of the rest.)

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

The Wronskian of n functions generalizes $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$:

$$W(y_1, y_2, \ldots, y_n)(t) = \det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ {y_1}'(t) & {y_2}'(t) & \cdots & {y_n}'(t) \\ {y_1}''(t) & {y_2}''(t) & \cdots & {y_n}''(t) \\ \cdots & \cdots & \cdots & \cdots \\ {y_1}^{(n-1)}(t) & {y_2}^{(n-1)}(t) & \cdots & {y_n}^{(n-1)}(t) \end{bmatrix}.$$

The theory of ODEs uses the following facts from linear algebra:
Suppose M is a square ($n \times n$) matrix, and $\mathbf{d}$ is a constant vector. Consider the equation $M\mathbf{c} = \mathbf{d}$ for the unknown vector $\mathbf{c}$. (This can be written as a system of n linear equations in the n unknowns $c_1, c_2, \ldots, c_n$.)

- If $\det(M) \neq 0$, then the equation has a unique solution, namely $\mathbf{c} = M^{-1}\mathbf{d}$.
- If $\det(M) = 0$, then the equation has either no solutions or an infinite number of solutions. Furthermore, the matrix M has no inverse.

Thus, we can solve the following system for c_1 and c_2 if $W(y_1, y_2)(t_0) \neq 0$, and there are either no solutions or infinitely many solutions if $W(y_1, y_2)(t_0) = 0$:

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$$