MAT 239 - Swift : The Logistic Equation

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right), \quad P(0)=P_0. \quad \left(\text{The positive parameters are } r, K.\right)$$

One can solve by Separation of variables, but it's a slog.

Instead, I'll use the procedure from Set 4, Problem 8.

let $u(t) = (P(t))^{-1}$, or $u = \frac{1}{P}$.

$$u'(t) = -P^{-2}\frac{dP}{dt} = -P^{-2}\left[rP\left(1-\frac{P}{K}\right)\right] = -\frac{r}{P}\left(1-\frac{P}{K}\right) = \frac{-r}{P} + \frac{r}{K}$$

or $\boxed{\frac{du}{dt} = -ru + \frac{r}{K}, \quad u(0) = \frac{1}{P_0}}$ Solve this by inspection!

$\frac{du}{dt} = -r\left(u - \frac{1}{K}\right)$, so the general solution is

$$u(t) = \frac{1}{K} + Ce^{-rt}$$

$$u(0) = \frac{1}{K} + C \stackrel{\text{set}}{=} \frac{1}{P_0}, \text{ so } C = \frac{1}{P_0} - \frac{1}{K}$$

$$u(t) = \frac{1}{K} + \left(\frac{1}{P_0} - \frac{1}{K}\right)e^{-rt}$$

Take reciprocal & simplify

$$P(t) = \frac{1}{\frac{1}{K} + \left(\frac{1}{P_0} - \frac{1}{K}\right)e^{-rt}} = \frac{P_0K}{P_0K\left[\frac{1}{K} + \left(\frac{1}{P_0} - \frac{1}{K}\right)e^{-rt}\right]}$$

$$\boxed{P(t) = \frac{P_0K}{P_0 + (K-P_0)e^{-rt}}}$$

This result is quoted in Set 6, Problem 5.

Another form of the answer is

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}}.$$

(Note that $P(t) \to K$ as $t \to \infty$.)