SHORTCUTS FOR SOLVING SOME INITIAL VALUE PROBLEMS

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ABSTRACT. We present an easily computed solution to the Initial Value Problem (IVP) for the Ordinary Differential Equation (ODE) $\dot{\mathbf{x}} = A\mathbf{x}$ with the initial condition $x(0) = \mathbf{x}_0$, when A is a 2×2 real matrix.

1. Introduction

Consider the initial value problem (IVP)

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

where $A \in \mathbb{R}^{2\times 2}$ and $\mathbf{x}_0 \in \mathbb{R}^2$. The solution is a function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ that satisfies the ordinary differential equation (ODE) $\mathbf{x}'(t) = A\mathbf{x}(t)$ and the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. The process of finding the solution is taught in undergraduate courses on ODEs. The solution is straightforward but messy, especially when the eigenvalues of A are repeated or complex. We present a simple solution for any 2×2 matrix A. Our solution does not require a computation of the eigenvectors of A.

2. Two-Dimensional ODEs

First we recall some results from the ODE course. The eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$, and for a 2×2 matrix they can be written in terms of the trace and determinant of A. The set of eigenvalues $\{\lambda_1, \lambda_2\}$ has the elements

$$\frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\mathrm{det}A}}{2} = \tfrac{\mathrm{tr}A}{2} \pm \sqrt{\left(\tfrac{\mathrm{tr}A}{2}\right)^2 - \mathrm{det}A}.$$

$$\frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}(A))^2 - 4\mathrm{det}(A)}}{2} = \tfrac{\mathrm{tr}(A)}{2} \pm \sqrt{\left(\tfrac{\mathrm{tr}(A)}{2}\right)^2 - \mathrm{det}(A)}.$$

The eigenvectors $\mathbf{v}_i \in \mathbb{R}^2$ are nonzero vectors that satisfy $(A - \lambda_i I)\mathbf{v}_i = \mathbf{0}$, for $i \in \{1, 2\}$. When the eigenvalues of A are real and distinct, the general solution to the ODE is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$, and the IVP is solved by replacing $c_1, c_2 \in \mathbb{R}$ with the solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}_0$. When the eigenvalues of A are complex the real and imaginary part of the complex-valued function $e^{\lambda_1}\mathbf{v}_1$ are two linearly independent solutions. When the eigenvalues are repeated a second linearly independent solution to the ODE is found in a process described at Paul's notes.

Eigenvectors are very important in science and engineering, and they give deep understanding of the phase portrait of the ODE. It is with mixed feelings that we present a methods of solving IVPs that do not use the eigenvectors at all. However, the ODE course is very recipe-based, giving algorithms on how to solve many classes of ODEs. The methods given here are so efficient that I believe they should be made available to students.

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In this section we present three propositions, which reflect the three possible signs of $(\mathrm{tr}A)^2 - 4\det A$. We start with $(\mathrm{tr}A)^2 - 4\det A < 0$, so the eigenvalues of A are complex.

Proposition 2.1. *Assume the eigenvalues of $A \in \mathbb{R}^{2\times 2}$ are $a \pm ib$, where $a, b \in \mathbb{R}$, with $b \neq 0$. Let $\mathbf{x}_0 \in \mathbb{R}^2$ and define $\mathbf{x}_1 := (A - aI)\mathbf{x}_0$. The solution to the IVP (1) is*

$$\mathbf{x}(t) = e^{at}\left(\cos(bt)\,\mathbf{x}_0 + \tfrac{\sin(bt)}{b}\,\mathbf{x}_1\right). \tag{2}$$

Proof. Let $\lambda_1 = a + ib$ be an eigenvalue of A with associated eigenvector $\mathbf{v}_1 = \mathbf{v}_R + i\mathbf{v}_I \in \mathbb{C}^2$, where $\mathbf{v}_R, \mathbf{v}_I \in \mathbb{R}^2$. The real and imaginary parts of the eigenvalue equation $A(\mathbf{v}_R + i\mathbf{v}_I) = (a+ib)(\mathbf{v}_R + i\mathbf{v}_I)$ are $A\mathbf{v}_R = a\mathbf{v}_R - b\mathbf{v}_I,$ and $A\mathbf{v}_I = a\mathbf{v}_I + b\mathbf{v}_R$. This implies that

$$(A - aI)\mathbf{v}_R = -b\mathbf{v}_I \text{ and } (A - aI)\mathbf{v}_I = b\mathbf{v}_R.$$

These two equations are not independent. Assume the second equation is true. Then

$$\mathbf{v}_R = \frac{1}{b}(A - aI)\mathbf{v}_I, \tag{3}$$

and the first equation is true since

$$(A - aI)\mathbf{v}_R + b\mathbf{v}_I = \frac{1}{b}(A - aI)^2\mathbf{v}_I + b\mathbf{v}_I = \frac{1}{b}\left((A - aI)^2 + b^2 I\right)\mathbf{v}_I = \mathbf{0}.$$

The last equal sign holds since every matrix satisfies its characteristic equation, which for A can be written as $(\lambda - a)^2 + b^2 = 0$.

While the eigenvalues are unique, the eigenvectors are not unique since any nonzero complex scalar multiple of an eigenvector is an eigenvector. This freedom can be used to our advantage. Let $\mathbf{v}_I = \mathbf{x}_0$. The first equation in (3) implies that $\mathbf{v}_R = \frac{1}{b}(A - aI)\mathbf{x}_0$. Thus $\mathbf{v}_1 = \frac{1}{b}\mathbf{x}_1 + i\mathbf{x}_0$, where $\mathbf{x}_1$ is defined in the statement of the proposition. With this choice of the eigenvector, one real-valued solution to the ODE is $\mathrm{Im}(e^{\lambda_1 t}\mathbf{v}_1) = \mathrm{Im}(e^{at}(\cos(bt) + i\sin(bt))(\frac{1}{b}\mathbf{x}_1 + i\mathbf{x}_0)) = e^{at}\big(\cos(bt)\mathbf{x}_0 + \frac{\sin(bt)}{b}\mathbf{x}_1\big)$. This is the expression given in Equation (2), and it satisfies the initial condition since $\cos(0) = 1$ and $\sin(0) = 0$. ☐

Remark 2.2. Note that the eigenvalues of A and the solution in Equation (2) are the same if we choose $b > 0$ or $b < 0$. We can choose $b > 0$ without loss of generality.

Example 2.3. Let $A = \begin{bmatrix} -1 & -2 \\ 5 & -3 \end{bmatrix}$. The eigenvalues are $-2 \pm 3i$, so $a = -2$ and $b = 3$ in the proposition, and $A - aI = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}$. The solution to the IVP with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is thus $\mathbf{x}(t) = e^{-2t}\left(\cos(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{\sin(3t)}{3}\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right)$.

Example 2.4. It is quite easy to find the general solution to an ODE by choosing a basis of initial conditions. For example, with the same matrix as in Example *pervious*, the solution with with $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\mathbf{x}(t) = e^{-2t}\left(\cos(3t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{\sin(3t)}{3}\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)$. The general solution is a linear combination of these two solutions,

$$\mathbf{x}(t) = e^{-2t}\begin{bmatrix} \cos(3t) + \frac{1}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) \end{bmatrix} c_1 + e^{-2t}\begin{bmatrix} -\frac{2}{3}\sin(3t) \\ \cos(3t) - \frac{1}{3}\sin 3t \end{bmatrix} c_2.$$

This is arguably the best way of writing the general solution, since the solution for the arbitrary initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{-2t} \begin{bmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin 3t \end{bmatrix} \mathbf{x}_0,$$

since the constants are $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_0$.

Next we consider when $(\mathrm{tr}A)^2 - 4\det A = 0$, so A has repeated eigenvalues which we denote as $\lambda = a, a$. The characteristic equation of A is $(\lambda - a)^2 = 0$, so $(A - aI)^2 = 0$. We say that A has algebraic multiplicity 2. It is possible that $A = aI$, in which case every nonzero vector is an eigenvector and A has geometric multiplicity 2. More commonly, $A \neq aI$, and there is a unique eigenvector up to scalar multiples and A has geometric multiplicity 1. In most undergraduate ODE classes these two cases are considered separately, but the following Proposition gives a unified solution.

Proposition 2.5. *Assume that the eigenvalue a of $A \in \mathbb{R}^{2\times 2}$ has algebraic multiplicity 2. Let $\mathbf{x}_0 \in \mathbb{R}^2$ and define $\mathbf{x}_1 := (A - aI)\mathbf{x}_0$. The solution to the IVP (1) is*

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 + t\mathbf{x}_1\right). \tag{4}$$

Proof. Note that $(A - aI)\mathbf{x}_1 = (A - aI)^2\mathbf{x}_0 = 0$. Assume Equation (4) holds. Then

$$\begin{aligned} A\mathbf{x}(t) - \mathbf{x}'(t) &= e^{at}A(\mathbf{x}_0 + t\mathbf{x}_1) - ae^{at}(\mathbf{x}_0 + t\mathbf{x}_1) - e^{at}\mathbf{x}_1 \\ &= e^{at}\left((A - aI)\mathbf{x}_0 + t(A - aI)\mathbf{x}_1 - \mathbf{x}_1\right) \\ &= e^{at}\left(\mathbf{x}_1 + t \cdot 0 - \mathbf{x}_1\right) \\ &= \mathbf{0}. \end{aligned}$$

Thus, Equation (4) is a solution to the ODE, and it clearly satisfies the initial condition. □

Finally, we turn to the case where $(\mathrm{tr}A)^2 - \det A > 0$, and the eigenvalues of A are distinct and real. This is the most straightforward case using traditional methods, but it still takes quite a bit of calculation so solve an initial value problem.

Proposition 2.6. *Assume the eigenvalues of $A \in \mathbb{R}^{2\times 2}$ are $a \pm b$, where $a, b \in \mathbb{R}$, with $b \neq 0$. Let $\mathbf{x}_0 \in \mathbb{R}^2$ and define $\mathbf{x}_1 := (A - aI)\mathbf{x}_0$. The solution to the IVP (1) is*

$$\mathbf{x}(t) = e^{at}\left(\cosh(bt)\,\mathbf{x}_0 + \tfrac{\sinh(bt)}{b}\,\mathbf{x}_1\right). \tag{5}$$

Proof. The definitions $\cosh(x) := (e^x + e^{-x})/2$ and $\sinh(x) := (e^x - e^{-x})/2$ show that

$$\mathbf{x}(t) = \left(\tfrac{1}{2}\mathbf{x}_0 + \tfrac{1}{2b}\mathbf{v}_0\right)e^{(a+b)t} + \left(\tfrac{1}{2}\mathbf{x}_0 - \tfrac{1}{2b}\mathbf{v}_0\right)e^{(a-b)t}.$$

Note that $\frac{1}{2}\mathbf{x}_0 + \frac{1}{2b}\mathbf{v}_0 = \frac{1}{2b}(b\mathbf{x}_0 + (A - aI)\mathbf{x}_0) = \frac{1}{2b}(A - (a-b)I)\mathbf{x}_0$. Thus $\frac{1}{2}\mathbf{x}_0 + \frac{1}{2b}\mathbf{v}_0$ is an eigenvector of A with eigenvalue $a + b$, since

$$(A - (a+b)I)(\tfrac{1}{2}\mathbf{x}_0 + \tfrac{1}{2b}\mathbf{v}_0) = \tfrac{1}{2b}(A - (a+b)I)(A - (a-b)I)\mathbf{x}_0 = \mathbf{0}.$$

The zero matrix multiplies $\mathbf{x}_0$ in that last equation, since the characteristic polynomial of A can be factored as $(\lambda - (a+b))(\lambda - (a-b)) = 0$, and a matrix satisfies its own characteristic polynomial. A similar argument shows that $\frac{1}{2}\mathbf{x}_0 - \frac{1}{2b}\mathbf{v}_0$ is an eigenvector of A with eigenvalue $a - b$. Thus Equation (5) is a solution to $\dot{\mathbf{x}} = A\mathbf{x}$, and the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is satisfied since $\cosh(0) = 1$ and $\sinh(0) = 0$. □

As an example, we compute the solution to $\dot{\mathbf{x}} = A\mathbf{x}$ for an arbitrary initial condition, and thus compute $\exp(At)$, for the matrix $A = [\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}]$. The eigenvalues, $\lambda = (5 \pm \sqrt{5})/2$, are as messy as they can be for a 2×2 matrix with integer entries. Following Proposition 5,

$$(A - aI)\mathbf{x}_0 = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_0 + y_0 \\ x_0 + \frac{1}{2}y_0 \end{bmatrix},$$

so the solution is

$$\mathbf{x}(t) = e^{5t/2} \left(\cosh(\sqrt{5}t/2) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{\sinh(\sqrt{5}t/2)}{\sqrt{5}} \begin{bmatrix} -x_0 + 2y_0 \\ 2x_0 + y_0 \end{bmatrix} \right).$$

The solution can be written as $\mathbf{x}(t) = \exp(At)\mathbf{x}_0$, so we have computed

$$e^{At} = e^{5t/2} \left(\cosh(\sqrt{5}t/2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\sinh(\sqrt{5}t/2)}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \right).$$

I plan to use actual numbers rather than x_0, y_0, and not talk about matrix exponentials until the later section, which includes the following (and the remark which mentions e^{At}.

3. Higher Dimensional IVPs

Projection of $\mathbf{x}_0$ onto eigenspaces is messy, and I do not have shortcuts for that.

The cosh and sinh solution does not really have an analog in higher dimensions that I know of.

However, we can get a very nice result for matrices with a single eigenvalue.

We can get a less clean result for a $2n \times 2n$ matrix with eigenvalues $a \pm ib$ with multiplicity n.

We start with the fundamental theorem of linear systems:

Proposition 3.1. *Consider* $\mathbf{x}_0 \in \mathbb{R}^n$ *and* $A \in \mathbb{R}^{n \times n}$. *Let* $\mathbf{x}_k := A^k \mathbf{x}_0$ *for* $k \in \mathbb{N} := \{1, 2, \ldots\}$. *The solution to the IVP (1) is*

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{x}_1 + \tfrac{t^2}{2!}\mathbf{x}_2 + \cdots + \tfrac{t^k}{k!}\mathbf{x}_k + \cdots. \tag{6}$$

Proof. Equation (6) satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and it satisfies the ODE: OTHER PROOFS USE $A\mathbf{x}(t) - \mathbf{x}'(t)$.

$$\begin{aligned} \mathbf{x}'(t) - A\mathbf{x}(t) &= \mathbf{x}_1 + t\mathbf{x}_2 + \tfrac{t^2}{2!}\mathbf{x}_3 + \cdots + \tfrac{t^k}{k!}\mathbf{x}_{k+1} + \cdots \\ &\quad - (A\mathbf{x}_0 + tA\mathbf{x}_1 + \tfrac{t^2}{2!}A\mathbf{x}_2 + \cdots + \tfrac{t^k}{k!}A\mathbf{x}_k + \cdots) \\ &= (\mathbf{x}_1 - A\mathbf{x}_0) + t(\mathbf{x}_2 - A\mathbf{x}_1) + \cdots + \tfrac{t^k}{k!}(\mathbf{x}_{k+1} - A\mathbf{x}_k) + \cdots \\ &= \mathbf{0}, \end{aligned}$$

since $\mathbf{x}_k = A\mathbf{x}_{k-1}$ for all $k \in \mathbb{N}$. The series converges, as shown in [**?**]. $\square$

Remark 3.2. The typical statement of the Fundamental Theorem of Linear Systems [**?**] is that the solution to (1) is $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, where the matrix exponential is defined in terms of its Taylor series:

$$e^{At} = I + tA + \tfrac{t^2}{2!}A^2 + \cdots + \tfrac{t^k}{k!}A^k + \cdots$$

We now present two similar propositions where the solution is given in terms of a finite series.

Proposition 3.3. *Assume that the eigenvalue a of $A \in \mathbb{R}^{n\times n}$ has algebraic multiplicity n. Let $\mathbf{x}_0 \in \mathbb{R}^n$ and define $\mathbf{x}_k := (A - aI)^k \mathbf{x}_0$ for $k \in \{1, 2, \ldots, n-1\}$. The solution to the IVP (1) is*

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 + t\mathbf{x}_1 + \tfrac{t^2}{2!}\mathbf{x}_2 + \tfrac{t^3}{3!}\mathbf{x}_3 + \ldots + \tfrac{t^{n-1}}{(n-1)!}\mathbf{x}_{n-1}\right). \tag{7}$$

Proof. The characteristic equation of A is $(A-aI)^n = 0$, so $(A-aI)^n = 0$ and $(A-aI)\mathbf{x}_{n-1} = (A-aI)^n\mathbf{x}_0 = \mathbf{0}$. Define $\mathbf{x}_k := (A-kI)^k\mathbf{x}_0$ for all $k \in \mathbb{N}$ and note that $\mathbf{x}_k = \mathbf{0}$ if $k \geq n$. With this definition, expression in Equation (7) can be written as an infinite series, which satisfies

$$\begin{aligned}
\mathbf{x}'(t) - A\mathbf{x}(t) &= ae^{at}\left(\mathbf{x}_0 + t\mathbf{x}_1 + \ldots + \tfrac{t^k}{k!}\mathbf{x}_k + \cdots\right) \\
&\quad + e^{at}\left(\mathbf{x}_1 + t\mathbf{x}_2 + \ldots + \tfrac{t^k}{k!}\mathbf{x}_{k+1} + \cdots\right) \\
&\quad - e^{at}\left(A\mathbf{x}_0 + tA\mathbf{x}_1 + \cdots + \tfrac{t^k}{k!}A\mathbf{x}_k + \cdots\right) \\
&= e^{at}\Big(\big(\mathbf{x}_1 - (A-aI)\mathbf{x}_0\big) + t\big(\mathbf{x}_2 - (A-aI)\mathbf{x}_1\big) \\
&\quad + \cdots + \tfrac{t^k}{k!}\big(\mathbf{x}_{k+1} - (A-aI)\mathbf{x}_k\big) + \cdots\Big) \\
&= \mathbf{0}.
\end{aligned}$$

The sum is equal to zero, since $\mathbf{x}_{k+1} - (A-aI)\mathbf{x}_k = \mathbf{0}$ for all integers $k \geq 0$. Thus, Equation (7) is a solution to the ODE, and it clearly satisfies the initial condition. □

Remark 3.4. A more sophisticated proof, along the lines of Perko, computes the matrix exponential e^{At}. Since aI commutes with $(A-aI)$, the exponential of the sum is the product of the exponentials, and

$$\begin{aligned}
e^{At} &= e^{aIt}e^{(A-aI)t} = (e^{at}I)e^{(A-aI)t} \\
&= e^{at}\left(I + t(A-aI) + \tfrac{t^2}{2!}(A-aI)^2 + \tfrac{t^3}{3!}(A-aI)^3 + \ldots + \tfrac{t^{n-1}}{(n-1)!}(A-aI)^{n-1}\right).
\end{aligned}$$

The Taylor series for $e^{(A-aI)t}$ terminates since $(A-aI)^n = 0$. With our definition of $\mathbf{x}_k$, the expression $e^{At}\mathbf{x}_0$ is the right-hand-side of Equation (7).

Finally, we can consider the case of repeated complex eigenvalues. This is not a case where the matrix exponential can be computed easily. The following proposition is probably known, but we are not aware of a statement of it. The proof extends the elementary techniques we have been using in the previous proofs of this section.

Proposition 3.5. *(Conjecture, actually) Assume that the eigenvalues of $A \in \mathbb{R}^{2n\times 2n}$ are $a \pm ib$ with $a, b \in \mathbb{R}$ and $b > 0$, and that both have algebraic multiplicity n. Let $\lambda_1 = a - ib$. Given $\mathbf{x}_0 \in \mathbb{R}^{2n}$, there is unique vector $\mathbf{y}_0 \in \mathbb{R}^{2n}$ such that $(A - \lambda_1 I)^n(\mathbf{x}_0 + i\mathbf{y}_0) = \mathbf{0}$. Define $\mathbf{x}_k, \mathbf{y}_k \in \mathbb{R}^{2n}$ for $k \in \{1, 2, \ldots, n-1\}$ by $\mathbf{x}_k + i\mathbf{y}_k = (A - \lambda_1 I)^k(\mathbf{x}_0 + i\mathbf{y}_0)$. The solution to the IVP (1) is*

$$\begin{aligned}
\mathbf{x}(t) &= e^{at}\cos(bt)\left(\mathbf{x}_0 + t\mathbf{x}_1 + \tfrac{t^2}{2!}\mathbf{x}_2 + \ldots + \tfrac{t^{n-1}}{(n-1)!}\mathbf{x}_{n-1}\right) \\
&\quad + e^{at}\sin(bt)\left(\mathbf{y}_0 + t\mathbf{y}_1 + \tfrac{t^2}{2!}\mathbf{y}_2 + \ldots + \tfrac{t^{n-1}}{(n-1)!}\mathbf{y}_{n-1}\right).
\end{aligned} \tag{8}$$

Proof. First we show the existence and uniqueness of $\mathbf{y}_0$. Define $A_R, A_I \in \mathbb{R}^{2n\times 2n}$ by $A_R + iA_I := (A - \lambda_1 I)^n$. Let $\lambda_2 = a + ib$, and note that the characteristic equation of A is

$(\lambda-\lambda_1)^n(\lambda-\lambda_2)^n=0$. Furthermore, $(A-\lambda_2 I)^n = A_R - iA_I$, so $(A_R+iA_I)(A_R-iA_I)=0$. This implies that

$$A_R^2 + A_I^2 = 0, \text{ and } A_R A_I = A_I A_R.$$

For a given $\mathbf{x}_0$, A_R and A_I we need to solve $(A_R+iA_I)(\mathbf{x}_0+i\mathbf{y}_0)=0$ for $\mathbf{y}_0$. This is equivalent to

$$A_R\mathbf{x}_0 - A_I\mathbf{y}_0 = \mathbf{0}, \text{ and } A_R\mathbf{y}_0 + A_I\mathbf{x}_0 = \mathbf{0}.$$

Assume that A_I is nonsingular. We can solve the first equation and get $\mathbf{y}_0 = A_I^{-1}A_R\mathbf{x}_0$. Then the second equation is satisfied since

$$A_R A_I^{-1} A_R \mathbf{x}_0 + A_I\mathbf{x}_0 = A_I^{-1}(A_R^2 + A_I^2) = 0.$$

I'm not sure how to prove that A_I is nonsingular. Try this: If A_I is singular, then there are infinitely many nonzero solutions in $\mathbb{R}^{2n}$ to $A_I\mathbf{v}=\mathbf{0}$. So what?

Following the proof of Proposition 3.3, define $\mathbf{x}_k + i\mathbf{y}_k = (A-\lambda_1 I)^k(\mathbf{x}_0+i\mathbf{y}_0)$ for all $k\in\mathbb{N}$. This implies the recurrence relation $\mathbf{x}_{k+1}+i\mathbf{y}_{k+1} = \big((A-aI)+ibI\big)(\mathbf{x}_k+i\mathbf{y}_k)$, which is equivalent to

$$\mathbf{x}_{k+1} = (A-aI)\mathbf{x}_k - b\mathbf{y}_k, \text{ and } \mathbf{y}_{k+1} = (A-aI)\mathbf{y}_k + b\mathbf{x}_k. \tag{9}$$

Note that $\mathbf{x}_k + i\mathbf{y}_k = \mathbf{0}$ for all $k\ge n$, so Equation (8) can be written with infinite series which satisfy

$$\begin{aligned}
\mathbf{x}'(t) - A\mathbf{x}(t) &= ae^{at}\cos(bt)\left(\mathbf{x}_0 + t\mathbf{x}_1 + \ldots + \tfrac{t^k}{k!}\mathbf{x}_k + \cdots\right)\\
&\quad + ae^{at}\sin(bt)\left(\mathbf{y}_0 + t\mathbf{y}_1 + \ldots + \tfrac{t^k}{k!}\mathbf{y}_k + \cdots\right)\\
&\quad - be^{at}\sin(bt)\left(\mathbf{x}_0 + t\mathbf{x}_1 + \ldots + \tfrac{t^k}{k!}\mathbf{x}_k + \cdots\right)\\
&\quad + be^{at}\cos(bt)\left(\mathbf{y}_0 + t\mathbf{y}_1 + \ldots + \tfrac{t^k}{k!}\mathbf{y}_k + \cdots\right)\\
&\quad + e^{at}\cos(bt)\left(\mathbf{x}_1 + t\mathbf{x}_2 + \ldots + \tfrac{t^k}{k!}\mathbf{x}_{k+1} + \cdots\right)\\
&\quad + e^{at}\sin(bt)\left(\mathbf{y}_1 + t\mathbf{y}_2 + \ldots + \tfrac{t^k}{k!}\mathbf{y}_{k+1} + \cdots\right)\\
&\quad - e^{at}\cos(bt)\left(A\mathbf{x}_0 + tA\mathbf{x}_1 + \cdots + \tfrac{t^k}{k!}A\mathbf{x}_k + \cdots\right)\\
&\quad - e^{at}\sin(bt)\left(A\mathbf{y}_0 + tA\mathbf{y}_1 + \cdots + \tfrac{t^k}{k!}A\mathbf{y}_k + \cdots\right)\\
&= e^{at}\cos(bt)\sum_{k=0}^{\infty}\tfrac{t^k}{k!}(a\mathbf{x}_k + b\mathbf{y}_k + \mathbf{x}_{k+1} - A\mathbf{x}_k)\\
&\quad + e^{at}\sin(bt)\sum_{k=0}^{\infty}\tfrac{t^k}{k!}(a\mathbf{y}_k - b\mathbf{x}_k + \mathbf{y}_{k+1} - A\mathbf{y}_k)\\
&= e^{at}\cos(bt)\sum_{k=0}^{\infty}\tfrac{t^k}{k!}\big(\mathbf{x}_{k+1} - (A-aI)\mathbf{x}_k + b\mathbf{y}_k\big)\\
&\quad + e^{at}\sin(bt)\sum_{k=0}^{\infty}\tfrac{t^k}{k!}\big(\mathbf{y}_{k+1} - (A-aI)\mathbf{y}_k - b\mathbf{x}_k\big)\\
&= \mathbf{0}.
\end{aligned}$$

The last equal sign is a consequence of Equation (9). The initial condition is satisfied, so Equation (8) solves the IVP (1). $\square$

Remark 3.6. I'm not sure if I should change the notation in the proposition about a complex conjugate eigenvalues with multiplicity 1. The $\mathbf{y}_0$ here was called $\frac{1}{b}\mathbf{x}_1$ in the previous proposition.

I'm not sure if I want the following formulas for the matrix exponential

If $\lambda = a \pm ib$ then

$$e^{At} = e^{at}\left(\cos(bt)I + \frac{\sin(bt)}{b}(A - aI)\right)$$

If $\lambda = a, a$ then

$$e^{At} = e^{at}\left(I + t(A - aI)\right)$$

If $\lambda = a \pm b$ then

$$e^{At} = e^{at}\left(\cosh(bt)I + \frac{\sinh(bt)}{b}(A - aI)\right)$$

This actually shows an easy way to get a formula that probably is very well known. If all of the eigenvalues of A are a, then $A = aI + (A - aI)$, and the "S+N" decomposition gives

$$e^{At} = e^{at}\left(I + t(A - aI) + \frac{t^2}{2}(A - aI)^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(A - aI)^{n-1}\right)$$

The eigenvalues of A satisfy $\det(\mathrm{A} - \lambda \mathrm{I}) = 0$, and the associated eigenvectors satisfy $A\mathbf{v} = \lambda\mathbf{v}$, or $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, is a solution to the ODE $\mathbf{x}' = A\mathbf{x}$, for real or complex eigenvalues λ. But complex eigenvalues have complex eigenvectors and $e^{\lambda t}\mathbf{v}$ is a complex-valued vector. With repeated eigenvalues, we cannot find two linearly independent eigenvectors.

Case 1. A has real, distinct eigenvalues, $\lambda_1 \neq \lambda_2$.
The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. The constants c_1 and c_2 are determined by the initial condition.

Case 2. A has complex eigenvalues, $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, with $b > 0$.
Compute $\mathbf{x}_1 = (A - aI)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 \cos(bt) + \mathbf{x}_1 \tfrac{1}{b}\sin(bt)\right)$$

Case 3. A has repeated, real eigenvalues, $\lambda_1 = \lambda_2$.
Compute $\mathbf{x}_1 = (A - \lambda_1 I)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{\lambda_1 t}\left(\mathbf{x}_0 + \mathbf{x}_1 t\right)$$

Note that cases 2 and 3 are actually easier computations. I don't think these formulas are in Paul's notes or our suggested textbook. If you see them somewhere on the web, send me the link!

In case 2, an eigenvector for λ_1 is $\mathbf{x}_0 - i\frac{1}{b}\mathbf{x}_1$, and this formula follows from the general solution given in the book and Paul's notes.

In case 3, if $\mathbf{x}_0$ is an eigenvector, then $\mathbf{x}_1 = \mathbf{0}$ and the solution is simply $\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{x}_0$. If $\mathbf{x}_0$ is not an eigenvector, then $\mathbf{x}_1 \neq \mathbf{0}$ is an eigenvector. The solution given follows from Paul's notes.

Note that case 3 is obtained from case 2 in the limit $b \to 0$, since $\lim_{b\to 0}\cos(bt) = 1$ and $\lim_{b\to 0}\frac{1}{b}\sin(bt) = t$.

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