MAT 239 (Differential Equations), Prof. Swift

Group work on Solving Certain ODEs by Inspection
Friends don't let friends do unnecessary integrals

key.

Work on this together with a group of 3 or 4 people. Turn in your own paper. This worksheet is worth 5 class points.

A common ODE in applications has the form $\frac{dy}{dt} = -2(y-3)$. (t is time.)
In the previous worksheet we used the recipe for solving linear first order ODEs to find the general solution to this ODE. This ODE is also separable, but it is very difficult to solve it as a separable ODE. You may remember Set 3, problem 4.

1. Write down the constant solution to this ODE by inspection.
$y(t) = 3$, for all t. We sometimes write simply $y = 3$.

2. Fill in the blanks, assuming C is an arbitrary constant.
$\frac{d}{dt}(3 + Ce^{-2t}) = Ce^{-2t}\cdot(-2)$ $\quad -2\left((3 + Ce^{-2t}) - 3\right) = -2Ce^{-2t}$

← same! →

3. Using the answers to question 2, write down the general solution to $\frac{dy}{dt} = -2(y-3)$. Then find the solution with $y(0) = 0$.

general solution: $\boxed{y = 3 + Ce^{-2t}}$

think this to verify: $\frac{dy}{dx} = Ce^{-2t}(-2)$, $-2(y-3) = -2(Ce^{-2t})$
yes, it's a solution for any C

Plug in I.C.

$0 = 3 + C\cancel{e^{-2\cdot 0}}^{1}$

$C = -3$. Thus $\boxed{y = 3 - 3e^{-2t} \text{ is the solution to the IVP}}$

4. Solve the IVP from my version of Set 3, problem 4, without doing any integrals.
Find the solution to the differential equation $\frac{dy}{dt} = 0.7(y - 150)$ if $y = 50$ when $t = 0$.

general solution: $y = 150 + Ce^{0.7t}$

IC $t=0, y=50$ $\quad 50 = 150 + C\cancel{e^{0.7\cdot 0}}^{1}$

$C = -100$

$\boxed{y = 150 - 100e^{0.7t}}$

"P is proportional to Q" means that $P = kQ$ for some constant of proportionality, k.

Newton's Law of Cooling says that the rate of change of the temperature of an object is proportional to the temperature difference between the object and its surroundings. Let $y(t)$ be the temperature of the object, and let A be the temperature of the surroundings (the Ambient temperature). Then Newton's Law of Cooling says

$$\frac{dy}{dt} = -k(y - A)$$

where k is an unknown *positive* constant of proportionality.

Note: $k>0$ gives $\frac{dy}{dt}<0$ if $y>A$, $\frac{dy}{dt}>0$ if $y<0$

Hot things cool down, cold things warm up!

5. Suppose a cup of cold water, starting at 40 degrees Fahrenheit is placed in a 70 degree room. Let $y(t)$ be the temperature of the water after t minutes. Write down the initial value problem for y, with an unknown constant k.

$$\frac{dy}{dt} = -k(y-70), \quad y(0)=40$$

6. Write down the general solution to the ODE by inspection, and solve the initial value problem. Your answer will be a function of both t and k.

The general solution is $y = 70 + Ce^{-kt}$

Plug in $y=40$, $t=0$: $40 = 70 + C\cancel{e^{-k\cdot 0}}$ ($=1$)

$40 = 70 + C$

$C = 40 - 70 = -30$

$$\boxed{y = 70 - 30e^{-kt}}$$

or $y(t) = 70 - 30e^{-kt}$

7. Suppose that the ~~temperate~~ temperature of the water is 55 degrees after 10 minutes. What is e^{-10k}? What is k?

$y(10) = 55$

Plug in $t=10$, $y=55$ & solve for k:

$55 = 70 - 30e^{-k\cdot 10}$

$-15 = -30e^{-10k}$

$e^{-10k} = \frac{-15}{-30} = \frac{1}{2}$

$\boxed{e^{-10k} = \frac{1}{2}}$

$-10k = \ln(\frac{1}{2}) = -\ln(2)$

$\boxed{k = \frac{\ln(2)}{10}}$

8. What is $e^{-20k} = (e^{-10k})^2$? What is the temperature of the water after 20 minutes? Give a simplified answer, without logs or exponentials.

$e^{-20k} = (\frac{1}{2})^2 = \frac{1}{4}$, so $y(20) = 70 - 30e^{-k\cdot 20} = 70 - 30\cdot\frac{1}{4} = 70 - 7.5$

$\boxed{y(20) = 62.5}$

$\boxed{\text{The temperature is } 62.5° \text{ after 20 minutes.}}$