MAT 239 (Differential Equations), Prof. Swift
Worksheet 13 on Exact ODEs

1. Show that the ODE $\big(y^2 + \cos(x)\big)dx + 2xy\,dy = 0$ is exact.

$\frac{\partial}{\partial y}(y^2 + \cos(x)) \stackrel{?}{=} \frac{\partial}{\partial x}(2xy) \quad 2y \stackrel{?}{=} 2y \quad \therefore$ ODE is exact ✓

2. That is, the ODE is really $dF := \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial x}dx = 0$ in disguise, and the general solution is $F(x, y) = C$.

Write down the two facts you know about F:

I. $\dfrac{\partial F}{\partial x} = y^2 + \cos(x)$ and II. $\dfrac{\partial F}{\partial y} = 2xy$.

Method A: We obtain the formula for $F(x, y)$ by finding 2 antiderivatives (that is, by doing 2 "partial integrals" with functions replacing the "$+C$"), and finding an $F(x, y)$ that satisfies both expressions.

Equation I says that $F(x, y) = xy^2 + \sin(x) \quad +g(y)$

Equation II says that $F(x, y) = xy^2 \quad +h(x)$

One choice of F, with no arbitrary constant, is $F(x, y) = xy^2 + \sin(x)$

Thus, the general solution to the ODE is $\boxed{xy^2 + \sin(x) = c}$

Also do method B: Take the expression for $F(x, y)$ with the $g(y)$ obtained from equation I and plug it into Equation II. Then solve for $g(y)$, which involves an arbitrary constant. Finally, get the formula for $F(x, y)$.

$F(x,y) = xy^2 + \sin(x) + g(y)$

$\dfrac{\partial F}{\partial y} = 2xy + g'(y) = 2xy$

$g'(y) = 0, \quad g(y) = c$

$F(x,y) = xy^2 + \sin(x) + \cancel{c} \to 0$

The solutin is $\boxed{xy^2 + \sin(x) = c}$