MAT 239 (Differential Equations), Prof. Swift
Worksheet 21, An Undamped, Undriven Oscillator

1. Suppose a ball stretches an ideal spring 6 inches. The ball is placed on the spring, where it oscillates up and down. Find the ODE for $y(t)$, the extension of the spring in feet beyond its equilibrium, after t seconds. Ignore friction and use $g = 32$ feet per second per second.

Hint: Let m be the mass of the ball. The spring constant k depends on m, but the ODE does not.

$F = kx$ is Hooke's law

$F = mg$, when $x = \frac{1}{2}$ (foot)

so $mg = k \cdot \frac{1}{2}$, and $k = 2mg$ → $m\frac{d^2y}{dt^2} = -2mgy$

$$\frac{d^2y}{dt^2} = -2gy = -64y$$

2. Write down the general solution to the ODE. It's OK to write down the solution by inspection.

$y'' = -64y$,

$r^2 = -64$

$r = \pm 8i$

$$y = C_1\cos(8t) + C_2\sin(8t)$$

3. Find the solution $y(t)$ to the ODE for arbitrary initial conditions $y(0) = y_0$ and $y'(0) = v_0$.

$y(0) = C_1\cos(0) + C_2\sin(0) = C_1 \overset{\text{set}}{=} y_0$ → $y = y_0\cos(8t) + \frac{v_0}{8}\sin(8t)$

$y'(t) = -8C_1\sin(8t) + 8C_2\cos(8t)$

$y'(0) = -8C_1 \cdot 0 + 8C_2 \cdot 1 = 8C_2 \overset{\text{set}}{=} v_0$ ⋯ ↑ $C_2 = \frac{v_0}{8}$

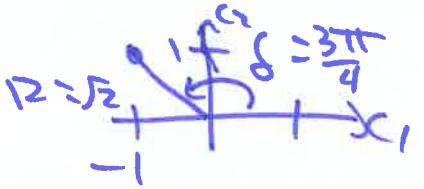
4. Find the particular solution to the ODE with the initial conditions $y(0) = -1$, $y'(0) = 8$.

Plug in $y_0 = -1$, $v_0 = 8$

$$y = -\cos(8t) + \frac{8}{8}\sin(8t)$$

$$y = -\cos(8t) + \sin(8t)$$

5. Write the solution to 4 in the $y(t) = R\cos(\omega_0 t - \delta)$ form. Draw a picture in the c_1-c_2 plane!



$C_1 = R\cos(\delta)$ } $R = \sqrt{2}$

$C_2 = R\sin(\delta)$ } $\delta = \frac{3\pi}{4}$

$$y(t) = \sqrt{2}\cos\left(8t - \frac{3\pi}{4}\right)$$

6. Sketch the solution to the IVP in problem 4 over at least two periods of the oscillation.

Period $= \frac{2\pi}{8} = \frac{\pi}{4}$

First max at $8t - \frac{3\pi}{4} = 0$

$8t = \frac{3\pi}{4}$

$t = \frac{3\pi}{32}$