MAT 239 (Differential Equations), Prof. Swift

Eigenvalue hack, and solving $\mathbf{x}' = A\mathbf{x}$ when $\lambda_1 \neq \lambda_2$ are real

The eigenvalues λ_1 and λ_2 of a matrix 2×2 matrix A, satisfy

$$\lambda_1 + \lambda_2 = T \text{ and } \lambda_1 \cdot \lambda_2 = D$$

where $T = \mathrm{Tr}(A)$ is the trace of A (the sum of the diagonal entries) and $D = \mathrm{Det}(A)$ is the determinant of A. That is enough to find the eigenvalues of A in many cases. If that doesn't work, use the fact that the characteristic equation of A is $\lambda^2 - T\lambda + D = 0$.

1. Use this hack to find the eigenvalues of the following matrices:

$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ $T = 3+4 = 7 = \lambda_1 + \lambda_2$, $D = 3 \cdot 4 - 1 \cdot 2 = 12 - 2 = 10 = \lambda_1 \lambda_2$, $\boxed{\lambda_1 = 2,\ \lambda_2 = 5}$

$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$ $T = -2$, $D = 0 - (-1) = 1$, $\lambda_1 + \lambda_2 = -2$, $\lambda_1 \cdot \lambda_2 = 1$, $\boxed{\lambda_1 = \lambda_2 = -1}$

$\begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$ See below

2. Fill in the first row so the eigenvalues are 3 and -2.

$\begin{bmatrix} a & b \\ 1 & 2 \end{bmatrix}$ $T = 3 + (-2) = \boxed{1 = a + 2} \Longrightarrow a = 1 - 2 = -1$

$D = 3(-2) = \boxed{-6 = 2a - b} \rightarrow b = 2a + 6 = -2 + 6 = 4$

$\begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix}$

3. Find the general solution to the system $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \mathbf{x}$, also written $\mathbf{x}' = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \mathbf{x}$.

$\lambda_1 = 2$, $\lambda_2 = 5$ From problem 1.

$\lambda_1 = 2$: $(A - 2I)\vec{v}_1 = \vec{0}$

let $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ $A - 2I = \begin{bmatrix} 3-2 & 2 \\ 1 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

So $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad a + 2b = 0$, $a + 2b = 0$ let $b = 1$, $a = -2$

$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is one choice.

$\lambda_2 = 5$ $A - 5I = \begin{bmatrix} 3-5 & 2 \\ 1 & 4-5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$

So $\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -2a + 2b = 0$, $a - b = 0$ $\therefore a = b$.

choose $a = b = 1$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The general solution is $\boxed{\vec{x}(t) = C_1 e^{2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$A = \begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}$ $T = -3 + 1 = -2$, $D = -3 + 8 = 5$

$\left.\begin{matrix} \lambda_1 + \lambda_2 = -2 \\ \lambda_1 \cdot \lambda_2 = 5 \end{matrix}\right\}$ No ~~real~~ integer solutions.

So $\lambda^2 - (-2)\lambda + 5 = 0$, $\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{(2)^2 - 4 \cdot 1 \cdot 5}}{2} = \dfrac{-2 \pm \sqrt{4 - 20}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = \boxed{-1 \pm 2i}$