MAT 239 (Differential Equations), Prof. Swift
The Swift Method for Repeated Eigenvalues, and Normal Modes

1. Use the Swift method to solve the IVP $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Note: Paul does this the hard way in example 1 of his notes on repeated eigenvalues.

$\lambda_1 + \lambda_2 = 7+3 = 10$

$\lambda_1 \cdot \lambda_2 = 7 \cdot 3 - (-4)(1) = 21 + 4 = 25$

$\lambda_1 = \lambda_2 = 5$. Case III

$\vec{x}_1 = (A - 5I)\vec{x}_0 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 4-5 \\ -8+10 \end{bmatrix}$

$\vec{x}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The Swift method says

$$\boxed{\vec{x}(t) = e^{5t}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} t\right)}$$

2. Find the general solution to $\frac{d^2\mathbf{x}}{dt^2} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}$.

Hint: Use the fact that if $A\mathbf{v} = -k^2\mathbf{v}$, then $\mathbf{x}(t) = (c_1 \cos(kt) + c_2 \sin(kt))\mathbf{v}$ is a solution to $\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x}$. This family of solutions is called a normal mode of the system.

Find eigenvalues and eigenvectors of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = (-2-\lambda)^2 - 1 = 0$

$\lambda_1 = -1, \quad \lambda_2 = -3$

$(\lambda+2)^2 - 1 = 0$

$(\lambda+2)^2 = 1$

$\lambda + 2 = \pm 1$

$\lambda = -2 \pm 1$

$\lambda_1 = -1$.

$A - \lambda_1 I = A + I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

$-a + b = 0 \therefore a = b$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is simplest choice.

$a - b = 0$

$\lambda_1 = -1, \quad k = 1$

$\lambda_2 = -3$:

$A + 3I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $\vec{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ satisfies $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$a + b = 0 \therefore a = -b$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$k_2 = \sqrt{3}$ $\quad \lambda_2 = -3$ is standard choice.

The general solution is

$$\boxed{\vec{x}(t) = (C_1 \cos(t) + C_2 \sin(t))\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (C_3 \cos(\sqrt{3}t) + C_4 \sin(\sqrt{3}t))\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Normal mode with $x_1 = x_2$

Normal mode with $x_1 = -x_2$