

MAT 239 (Differential Equations), Prof. Swift
Group work on Solving Certain ODEs by Inspection
Friends don't let friends do unnecessary integrals

Work on this together with a group of 3 or 4 people. No calculators or internet devices are allowed. Turn in one sheet for each group, and keep your own paper. Put your names at the top of the sheet your turn in. Everybody must write the answer to at least one of these problems, and put their initials next to the work. Failure to follow these rules will lose points.

This worksheet is worth 5 class points.

I've put bonus facts in with red.

A common ODE in applications has the form $\frac{dy}{dt} = -2(y - 3)$. (t is time.)

In the previous worksheet we used the recipe for solving linear first order ODEs to find the general solution to this ODE. This ODE is also separable, but it is very difficult to solve it as a separable ODE. You may remember Set 3, problem 4.

1. Write down the **constant** solution to this ODE by inspection.

$y(t) = 3$, for all t . We sometimes write simply $y = 3$.

2. Fill in the blanks with the correct function of t , assuming C is a constant.

$$\frac{d}{dt}(3 + Ce^{-2t}) = 0 + Ce^{-2t}(-2); \quad -2((3 + Ce^{-2t}) - 3) = -2(Ce^{-2t})$$

$$= -2Ce^{-2t}$$

same fn. of t in both blanks

3. (a) Using the answers to question 2, write down the general solution to

$\frac{dy}{dt} = -2(y - 3)$ without doing any integrals. That is, solve the ODE by inspection.

$$y(t) = 3 + Ce^{-2t}$$

(b) Solve the initial value problem (IVP) $\frac{dy}{dt} = -2(y - 3)$, $y(0) = 0$.

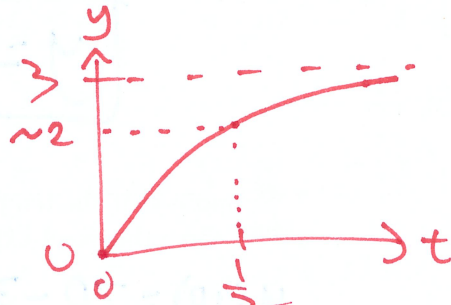
$$y(t) = 3 + Ce^{-2t}$$

$$y(0) = 3 + Ce^{-2 \cdot 0} = 0 \quad \text{set } = 0$$

$$e^0 = 1 \quad 3 + C = 0$$

$$C = -3$$

$$\text{so } \boxed{y(t) = 3 - 3e^{-2t}}$$



$$y(\frac{1}{2}) = 3 - 3e^{-1} \approx 3 - 1 = 2.$$

4. Solve the IVP from my version of Set 3, problem 4, without doing any integrals.

Find the solution to the differential equation $\frac{dy}{dt} = 0.7(y - 150)$ if $y = 50$ when $t = 0$.

I will evaluate $e^0 = 1$ in my head from now on.

$$y = 150 + Ce^{0.7t}$$

$$y(0) = 150 + C = 50 \quad \text{set } = 50$$

$$C = -100$$

I put a "-" sign here by mistake and then fixed it.

That is, \rightarrow

$$\text{so } \boxed{y = 150 - 100e^{+0.7t}}$$

$$\boxed{y = 150 - 100e^{0.7t}}$$

"P is proportional to Q" means that $P = kQ$ for some constant of proportionality, k .

Newton's Law of Cooling says that the rate of change of the temperature of an object is proportional to the temperature difference between the object and its surroundings. Let $y(t)$ be the temperature of the object, and let A be the temperature of the surroundings (the Ambient temperature). Then Newton's Law of Cooling says

$$\frac{dy}{dt} = -k(y - A)$$

where k is an unknown *positive* constant of proportionality.

5. Suppose a cup of cold water, starting at 40 degrees Fahrenheit is placed in a 70 degree room. Let $y(t)$ be the temperature of the water, in degrees Fahrenheit, after t minutes. Write down the initial value problem for y , with an unknown constant k . **Just give an ODE and an initial condition.** (IVP = ODE + IC.)

$$\frac{dy}{dt} = -k(y - 70), \quad y(0) = 40$$

6. Write down the general solution to the ODE by inspection, and then solve the initial value problem. Your answer will be a function of both t and k .

$$y = y(t) = 70 + C e^{-kt}$$

$$y(0) = 70 + C \stackrel{\text{set}}{=} 40$$

$$C = 40 - 70 = -30$$

$$y = 70 - 30e^{-kt}$$

know this!
 $\ln\left(\frac{1}{2}\right) = \ln(2^{-1}) = -\ln(2)$

7. Suppose that the temperature of the water is 55 degrees after 10 minutes. What is e^{-10k} ? Give the exact expression for k that you could approximate with a calculator.

$$y(10) = 70 - 30e^{-k \cdot 10} = 55$$

$$70 - 55 = 30e^{-10k}$$

$$\frac{15}{30} = e^{-k \cdot 10}, \quad e^{-10k} = \frac{1}{2}$$

Take ln of both sides
 $-10k = \ln\left(\frac{1}{2}\right)$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-10} = \frac{\ln(2)}{10}$$

8. (a) What is $e^{-20k} = (e^{-10k})^2$? (Simplify these as much as possible.)

$$e^{-20k} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

either one
 This is not relevant to problem 8.

(b) Using your expression for e^{-20k} , find the temperature of the water after 20 minutes. Give a decimal answer.

$$y(20) = 70 - 30e^{-20k} = 70 - 30 \cdot \frac{1}{4} = 70 - \frac{15}{2} = 70 - 7.5$$

$$y(20) = 62.5$$

$$e^{-20k} = \frac{1}{4}$$

Temp. diff. decreases by $\frac{1}{2}$ every 10 min

$$e^{-kt}$$

$$= e^{-10k \cdot \frac{t}{10}}$$

$$= (e^{-10k})^{\frac{t}{10}}$$

$$= \left(\frac{1}{2}\right)^{\frac{t}{10}}$$

Bonus

$$\text{Note: } y(t) = 70 - 30 \left(\frac{1}{2}\right)^{\frac{t}{10}}$$