MAT 239 (Differential Equations), Prof. Swift
Worksheet 16, on the Most Beautiful Fundamental Solution Set.

The ODE $y''+p(x)y'+q(x)y=0$ can be written as $L[y]=0$, where $L=D^2+p(x)D+q(x)$. Assume that p and q are continuous at 0. We don't know p and q, so we have no hope of actually writing down the formula for any solutions other than $y(x)=0$.

Let $y_1(x)$ be the solution to the IVP $L[y]=0, \quad y(0)=1,\ y'(0)=0$. Write down what this tells you about the function $y_1(x)$. (Recall day one: What is a DE? What is a solution to a DE?)

$L[y_1(t)]=0,\ y_1(0)=1,\ y_1'(0)=0$

Let $y_2(x)$ be the solution to the IVP $L[y]=0, \quad y(0)=0,\ y'(0)=1$. What do you know about y_2?

$L[y_2(t)]=0,\ y_2(0)=0,\ y_2'(0)=1$

Use the properties of linear operators to show that $L[c_1y_1(t)+c_2y_2(t)]=0$ for all t.

$$L[c_1y_1(t)+c_2y_2(t)] = c_1L[y_1(t)] + c_2L[y_2(t)]$$
$$= c_1\cdot 0 + c_2\cdot 0$$
$$= 0 \ \checkmark$$

You have just proved the superposition principle. If y_1 and y_2 are solutions to a linear homogeneous ODE, then $y=c_1y_1(t)+c_2y_2(t)$ is a solution to that same ODE for any real numbers c_1 and c_2. This is the general solution. Find the solution to $L[y]=0, \ \ y(0)=y_0,\ y'(0)=v_0$ for any y_0 and v_0.

$y = y_0\, y_1(t) + v_0\, y_2(t)$

Let's put some flesh on those bones. This page has only one ODE, but different ICs.

2. Solve the IVP $y'' - 4y = 0$, $y(0) = 1, y'(0) = 0$. Call this solution $y_1(x)$.

$r^2 - 4 = 0$
$r^2 = 4$
$r = \pm 2$

$y = C_1 e^{2x} + C_2 e^{-2x}$
$y' = 2C_1 e^{2x} - 2C_2 e^{-2x}$

$\boxed{\begin{array}{l} C_1 + C_2 = 1 \\ 2C_1 - 2C_2 = 0 \end{array}}$

$C_1 = C_2 = \frac{1}{2}$

$\boxed{y_1 = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}}$

3. Solve the IVP $y'' - 4y = 0$, $y(0) = 0, y'(0) = 1$. Call this solution $y_2(x)$.

same general solution, so

$C_1 + C_2 = 0 \therefore C_2 = -C_1$
$2C_1 - 2C_2 = 1$ $\quad 2C_1 - 2(-C_1) = 1$
$4C_1 = 1 \therefore C_1 = \frac{1}{4}, C_2 = -\frac{1}{4}$

$\boxed{y_2 = \frac{1}{4} e^{2x} - \frac{1}{4} e^{-2x}}$

4. Without doing any more work, solve these IVPS:

Solve the IVP $y'' - 4y = 0$, $y(0) = 3, y'(0) = 2$.

$\boxed{y = 3\left(\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}\right) + 2\left(\frac{1}{4} e^{2x} - \frac{1}{4} e^{-2x}\right)}$

Solve the IVP $y'' - 4y = 0$, $y(0) = -2, y'(0) = 1$.

$\boxed{y = -2\left(\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}\right) + \left(\frac{1}{4} e^{2x} - \frac{1}{4} e^{-2x}\right)}$

Solve the IVP $y'' - 4y = 0$, $y(0) = 0, y'(0) = -2$.

$\boxed{y = 2\left(\frac{1}{4} e^{2x} - \frac{1}{4} e^{-2x}\right)}$