MAT 239 (Differential Equations), Prof Swift
Worksheet 26, Series Solution

1. Find the recurrence relation for the ODE

$$y' + 2y = 0.$$

(The series solution is $y = \sum_{n=0}^{\infty} c_n x^n$. $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$
Plug this into the ODE and get c_{n+1} = something involving c_n and n.)

2. If you have ~~to~~ time, <u>solve</u> the recurrence relation. That is, get c_n as a function of n.

$$\underbrace{\sum_{n=1}^{\infty} n c_n x^{n-1}}_{y'} + \underbrace{2\sum_{n=0}^{\infty} c_n x^n}_{2y} = 0,$$

let $m = n-1$

so $n = m+1$, $n=1 \Rightarrow m=0$

$$\sum_{m=0}^{\infty} (m+1) c_{m+1} x^m + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

replace m with n, and move 2 in.

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

The ODE is now $\sum_{n=0}^{\infty}\left[(n+1)c_{n+1}+2c_n\right]x^n=0$ for all x.

So $[\ ]=0$ for all n. $\cancel{(n+1)c_{n+1}}\ (n+1)c_{n+1}+2c_n=0$ for $n=0,1,2,\ldots$

$$\boxed{c_{n+1}=\frac{-2c_n}{n+1} \text{ for } n=0,1,2,\ldots}$$

The recurrence relation.

2. Solve the recurrence relation.

c_0 is arbitrary

$n=0$: $c_1=\frac{-2c_0}{1}$

$n=1$: $c_2=\frac{-2c_1}{2}=\frac{(-2)^2c_0}{2\cdot1}$

$n=2$: $c_3=\frac{-2c_2}{3}=\frac{(-2)^3c_0}{3\cdot2\cdot1}$

$n=3$: $c_4=\frac{-2c_3}{4}=\frac{(-2)^4c_0}{4\cdot3\cdot2}$

In general $\boxed{c_n=\frac{(-2)^nc_0}{n!}}$

Plug this into series to find solution

$$y=\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}\frac{(-2)^nc_0x^n}{n!}$$

$$=c_0\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}.$$

Recognize Taylor series!

$$\boxed{y=c_0e^{-2x}}$$

we can solve ODE by inspection to get this same solution!