1. (a) Find the general solution to the ODE.

Linear!

$$\frac{dy}{dx} = -\frac{3y}{x} + 5x$$

$x=0$ is a bad place.

(b) Find the solution to this ODE with initial condition $y(1) = 0$.

(a) $\frac{dy}{dx} + \frac{3}{x}y = 5x$ solution on ~~$(-\infty,0)$~~ or on $(0,\infty)$ ✓

$p(x) = \frac{3}{x}$, $g(x) = 5x$

$\mu(x) = \exp\left(\int \frac{3}{x}\,dx\right) = \exp(3\ln|x| + \cancel{C}^{0})$

$= \exp(\ln(|x|^3)) = |x|^3$

$\mu(x) = x^3$ is good for $x>0$.

mult all terms by x^3

$x^3\frac{dy}{dx} + 3x^2y = 5x^4$

✓ $\frac{d}{dx}(x^3y) = 5x^4$

$x^3y = \int 5x^4\,dx = x^5 + C$

(a) $\boxed{y = \frac{x^5}{x^3} + \frac{C}{x^3} = x^2 + \frac{C}{x^3}}$ general solution

(b) Plug in $y(1)=0$

$y(1) = 1^2 + \frac{C}{1^3} = 1 + C \overset{\text{set}}{=} 0$

$C = -1$

So $\boxed{y(x) = x^2 - \frac{1}{x^3}}$ is the particular solution.

(Solution is valid on $(0,\infty)$, but I didn't ask for this.)

Blue version

2. (a) Find the solution to the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2}.$$

Separable.

(b) Find the interval of existence for the solution you found in part (a).

$$\frac{dy}{y^2} = dt$$

$$\int y^{-2}\,dy = \int dt$$

$$\frac{y^{-1}}{-1} = t + c$$

$$y^{-1} = -t - c$$

$$\frac{1}{y} = -t - c$$

$$y = \frac{1}{-t-c}$$

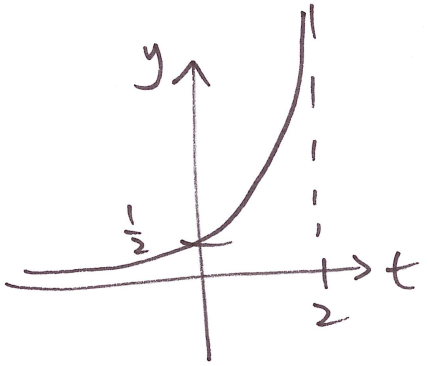
$$y(0) = \frac{1}{-0-c} = \frac{1}{2}, \text{ or } \frac{1}{-c} = \frac{1}{2} \therefore c = -2$$

$$y = \frac{1}{-t-(-2)} = \frac{1}{-t+2} = \frac{1}{2-t}$$

$$\boxed{y = \frac{1}{2-t}}$$

Solution "blows up" as $t \to 2$

The interval of existance is $\boxed{t<2}$, or $\boxed{(-\infty, 2)}$



3. Find the general solution for the first two ODEs, and find the particular solution to the last four parts, which are IVPs. You do not need to show your work on these because they can be solved by inspection.
Note that each of these problems has y as the dependent variable and t as the independent variable.

(a) $\frac{dy}{dt} = 2y$ $y = Ce^{2t}$

Note: $y = e^{2t+C}$ is NOT the general solution. It does NOT include $y=0$ or $y=-e^{2t}$ etc.

$y = e^{2t+C} = e^C \cdot e^{2t}$
But $e^C > 0$ for all C!

(b) $\frac{dy}{dt} = -3(y-2)$ $y = 2 + Ce^{-3t}$

(c) $\frac{dy}{dt} = e^t, \quad y(0) = 0$ $y = e^t + C$
$0 = e^0 + C \quad \therefore C = -1$
$y = e^t - 1$

(d) $\frac{dy}{dt} = 2y - 8, \quad y(0) = 3$ $= 2(y-4)$ $y = 4 + Ce^{2t}$
$3 = 4 + Ce^0 = 4 + C \quad \therefore C = -1$
$y = 4 - e^{2t}$ $y = 4 - e^{2t}$

(e) $\frac{dy}{dt} = k(y-A), \quad y(0) = 0$ $y = A + Ce^{kt}$
$y(0) = A + Ce^0 \overset{\text{set}}{=} 0$
$A + C = 0$, so $C = -A$
$y = A - Ae^{kt}$

(f) $\frac{dy}{dt} = \ln(y)\sin(t+y), \quad y(0) = 1$
$y = 1$, or $y(t) = 1$ (since $\ln(1) = 0$ and $\frac{d}{dt}(1) = 0$.)

Always look for the constant solution:
$y(t) = A, \; y'(t) = 0 = \ln(A)\sin(t+A)$
$\ln(A) = 0 \Rightarrow A = 1.$

$0 = \ln(1)\sin(t+1)$
$0 \checkmark 0$

4. Find the particular solution to

$$\frac{dy}{dx} = -\frac{2x+y^2}{2xy+5y^4}, \quad y(0)=1$$

NOT linear
NOT separable
Put into differential form and hope it's exact.

$(2xy+5y^4)dy = -(2x+y^2)dx$

$\boxed{(2x+y^2)dx + (2xy+5y^4)dy = 0}$ Differential Form. Essential

Test for exactness $\frac{\partial}{\partial y}(2x+y^2) \stackrel{?}{=} \frac{\partial}{\partial x}(2xy+5y^4)$

$2y \stackrel{\checkmark}{=} 2y$ $\therefore$ ODE is exact.

So the ODE is $F_x dx + F_y dy = 0$ in disguise, and the general solution is $F(x,y) = C$.

$\boxed{\begin{aligned} &\text{I.} \quad F_x = 2x+y^2 \\ &\text{II} \quad F_y = 2xy+5y^4 \end{aligned}}$ Super-Important. Write these down.

I says $F(x,y) = \int (2x+y^2)dx = x^2 + xy^2 + g(y)$

II says $F(x,y) = \int (2xy+5y^4)dy = xy^2 + y^5 + h(x)$

Both are true for $F(x,y) = x^2 + xy^2 + y^5 + \cancel{C}^{\,0}$

Method B

~~Alternative~~ Alternative method. Plug partial answer into II.

$\frac{\partial}{\partial y}(x^2 + xy^2 + g(y)) = 2xy + 5y^4$

$0 + 2xy + g'(y) = 2xy + 5y^4$, so $g'(y) = 5y^4$, and $g(y) = y^5 + \cancel{C}^{\,0}$

So again, $F(x,y) = x^2 + xy^2 + y^5$

The general solution is $\boxed{x^2 + xy^2 + y^5 = C}$

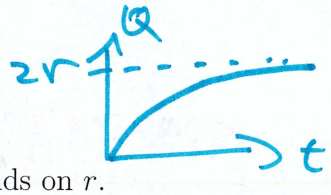
NOT soln. $F(x,y) = x^2 + xy^2 + y^5$

The general solution is NOT $y = F(x,y)$, or $y = x^2 + xy^2 + x^5$ NOT soln.

$F(0,1) = 0^2 + 0\cdot 1^2 + 1^5 = 1$, so the particular solution to the IVP is $\boxed{x^2 + xy^2 + y^5 = 1}$

5. Let $Q(t)$ be the mass (in kilograms) of radioactive material in a nuclear reactor, as a function of time t, measured in days. The material is being created at an unknown but constant rate of r kilograms per day, and it is subject to radioactive decay with a known half life. The mass Q obeys the initial value problem

$$\frac{dQ}{dt} = -\frac{1}{2}Q + r, \quad Q(0) = 0.$$



(a) Write down, by inspection if possible, the solution $Q(t)$, which depends on r.

$\frac{dQ}{dt} = -\frac{1}{2}(Q + \frac{r}{(-\frac{1}{2})}) = -\frac{1}{2}(Q - 2r)$

gen. soln: $Q(t) = 2r + Ce^{-\frac{1}{2}t}$: $Q(0)=0$ gives $C = -2r$

$\boxed{Q(t) = 2r - 2re^{-\frac{1}{2}t}}$ or $\boxed{Q(t) = 2r(1 - e^{-\frac{t}{2}})}$

(b) After waiting several months, it is found that there is approximately 10 kilograms of material. (That is, $\lim_{t\to\infty} Q(t) = 10$.) What is the rate r at which the radioactive material is being produced?

$\lim_{t\to\infty} e^{-\frac{1}{2}t} = 0$, so $\lim_{t\to\infty} Q(t) = 2r - 0 = 2r$

Set $2r = 10$, so $\boxed{r = 5}$

6. Recall that Euler's method is $x_{n+1} = x_n + h$, $y_{n+1} = y_n + hf(x_n, y_n)$.

Do two steps of Euler's method for the IVP $\frac{dy}{dx} = xy^2$, $y(0) = 2$, with step size $h = 3$. Box your final answer, which has the form $y(6) \approx y_2 = \cdots$.

Make a table

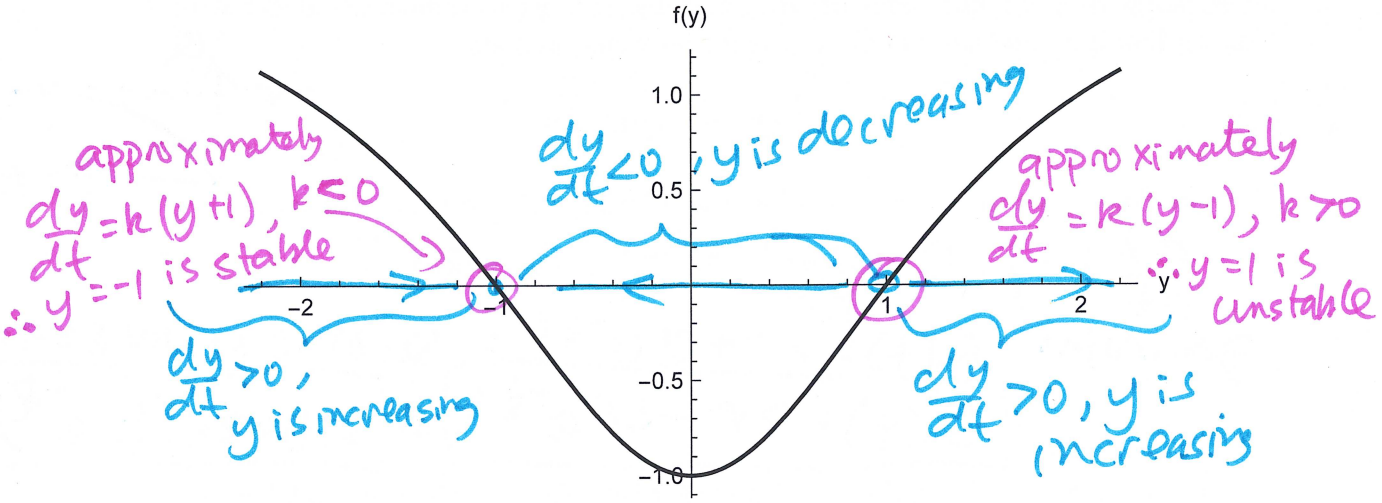
n	x_n	y_n
0	0	2
1	3	$2 + 3(0 \cdot 2^2) = 2$
2	6	$2 + 3(3 \cdot 2^2) = 2 + 9 \cdot 4 = 38$

So $\boxed{y(6) \approx y_2 = 38}$

or, we can skip the "n":

x	y
0	2
3	$2 + 3(0 \cdot 2^2) = 2$
6	$2 + 3(3 \cdot 2^2) = 38$

7. Consider the ODE $\frac{dy}{dt} = f(y)$, where the function f is plotted below:

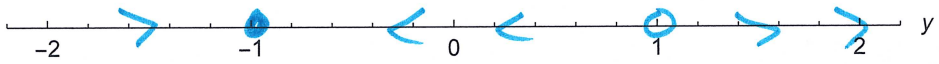


(a) Write down the equilibrium solutions, also known as constant solutions.

Solve $f(y)=0$ for y: $\boxed{y=-1 \text{ and } y=1}$

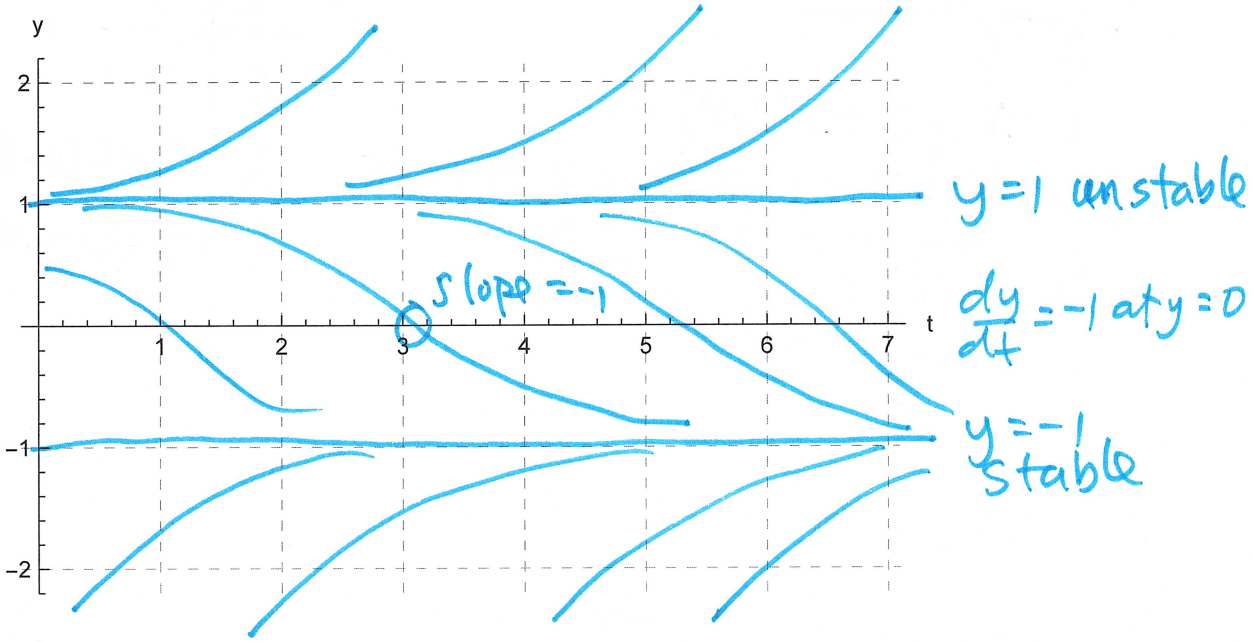
Note: "1" is not a solution. The function "$y=1$" is a solution

(b) Sketch the phase portrait of the ODE on the y-axis shown below.



The phase portrait shows the direction of motion of a particle with velocity $f(y)$ at position y.

(c) Sketch several solutions to the ODE in the t-y plane below. Be sure to draw the equilibrium solutions, and indicate if they are a stable equilibrium or an unstable equilibrium. Furthermore, draw the slope of the solution curves accurately as they cross the t axis.



Blue version