1. (a) Find the solution to the initial value problem

$$\frac{dy}{dt} = \frac{y^3}{2}, \quad y(0) = 1.$$

Separable!

(b) Find the interval of existence for the solution you found in part (a).

an interval is not something like $(-\infty, 1) \cup (1, \infty)$

$$\frac{dy}{y^3} = \frac{1}{2}dt$$

$$\int y^{-3}dy = \frac{1}{2}\int dt$$

$$\frac{y^{-2}}{-2} = \frac{1}{2}t + c$$

mult by (-2)

$$y^{-2} = -t + c \quad (\text{new } c)$$

$$\frac{1}{y^2} = -t + c$$

Plug in $y(0) = 1$

$$\frac{1}{1^2} = -0 + c$$

$$1 = c$$

So $\frac{1}{y^2} = -t + 1 \; (= 1 - t)$

$$y^2 = \frac{1}{1-t}$$

$$y = \pm\sqrt{\frac{1}{1-t}}$$ choose $+$ since $y(0) = +1$

$$\boxed{y = \frac{1}{\sqrt{1-t}}} \text{ or } \boxed{y(t) = \sqrt{\frac{1}{1-t}}}$$

(b) the interval is when $1 - t > 0$, or $\boxed{t < 1}$. or also written as $\boxed{(-\infty, 1)}$

Red version

2. (a) Find the general solution to the ODE.

$$\frac{dy}{dx} = \frac{-2y}{x} + 4x$$

Linear

(b) Find the solution to this ODE with initial condition $y(1) = 0$.

Put in standard form for a linear ODE

$$\frac{dy}{dx} + \frac{2}{x}y = 4x$$

$p(x) = \frac{2}{x}$, $g(x) = 4x$

$$\mu(x) = \exp\left(\int \frac{2}{x}dx\right) = \exp(2\ln|x| + c^{20})$$

$$= e^{2\ln|x|} = e^{\ln(|x|^2)} = e^{\ln(x^2)} = x^2$$

use $a\ln(b) = \ln(b^a)$

or, use $e^{2\ln|x|} = (e^{\ln|x|})^2 = |x|^2 = x^2$.

$(x^2)^3 = x^{2\cdot 3}$

multiply both sides of standard form by x^2

$$x^2\frac{dy}{dx} + 2xy = 4x^3$$

Check that the left hand side is the same!

$$\frac{d}{dx}(x^2y) = \left(\frac{d}{dx}(x^2)\right)\cdot y + x^2\frac{dy}{dx}$$

$$= 2xy + x^2\frac{dy}{dx}$$

yes!

$$\frac{d}{dx}(x^2y) = 4x^3$$

So $x^2y = \int 4x^3dx = x^4 + c$

so $y = \frac{x^4+c}{x^2} = x^2 + \frac{c}{x^2}$

the general solution.

(b) $y(1) = 1^2 + \frac{c}{1^2} = 1 + c \overset{\text{set}}{=} 0 \quad \therefore c = -1$

$y = \frac{x^4-1}{x^2} = x^2 - \frac{1}{x^2}$

either form is OK.

Red version

3. Let $Q(t)$ be the mass (in kilograms) of radioactive material in a nuclear reactor, as a function of time t, measured in days. The material is being created at an unknown but constant rate of r kilograms per day, and it is subject to radioactive decay with a known half life. The mass Q obeys the initial value problem

$$\frac{dQ}{dt} = -2Q + r, \quad Q(0) = 0.$$

Correct!

(a) Write down, by inspection if possible, the solution $Q(t)$, which depends on r.

$\frac{dQ}{dt} = -2\left(Q + \frac{r}{-2}\right) = -2\left(Q - \frac{r}{2}\right)$

gen. soln. $Q(t) = \frac{r}{2} + Ce^{-2t}$

$Q(0) = \frac{r}{2} + Ce^{0} = \frac{r}{2} + C \overset{\text{Set}}{=} 0 \quad \therefore \quad C = -\frac{r}{2}$

$Q(t) = \frac{r}{2} - \frac{r}{2}e^{-2t}$

So $Q(t) = \frac{r}{2} - \frac{r}{2}e^{-\frac{r}{2}t}$

This graph might help in Part (b)



$\lim_{t\to\infty} Q(t) = \frac{r}{2}$.

(b) After several months, it is found that there is approximately 5 kilograms of material. (That is, $\lim_{t\to\infty} Q(t) = 5$.) What is the rate r at which the radioactive material is being produced by the nuclear reactor?

$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(\frac{r}{2} - \frac{r}{2}e^{-\frac{r}{2}t}\right) = \frac{r}{2} \overset{\text{Set}}{=} 5$

So $r = 10$

Note: $\lim_{t\to\infty} e^{-\frac{r}{2}t} = 0$.

4. Recall that Euler's method is $x_{n+1} = x_n + h$, $y_{n+1} = y_n + hf(x_n, y_n)$.

Do two steps of Euler's method for the IVP $\frac{dy}{dx} = xy^2$, $y(0) = 3$, with step size $h = 2$. Box your final answer, which has the form $y(4) \approx y_2 = \cdots$.

n	x_n	y_n
0	0	3
1	2	$3+2(0\cdot 3^2) = 3$
2	4	$3+2(2\cdot 3^2) = 3+36 = 39$

So $y_2 = 39$

or simply

x	y
0	3
2	$3+2(0\cdot 3^2) = 3$
4	$3+2(2\cdot 3^2) = 39$

So $y(4) \approx y_2 = 39$

5. Find the general solution for the first two ODEs, and find the particular solution to the last four parts, which are IVPs. You do not need to show your work on these because they can be solved by inspection.
Note that each of these problems has y as the dependent variable and t as the independent variable.

(a) $\frac{dy}{dt} = y$ $y = ce^t$

Note: $y = e^{t+c} = e^c \cdot e^t$ is not the general solution, since $e^c > 0$ for all c. No c gives the solution $y = 0$, or $y = -e^t$.

(b) $\frac{dy}{dt} = -2(y-3)$ $y = 3 + ce^{-2t}$

(c) $\frac{dy}{dt} = e^t, \quad y(0) = 0$ $y(t) = e^t + c$

$y(0) = e^0 + c = 0 \therefore c = -1$

$\boxed{y = e^t - 1}$

(d) $\frac{dy}{dt} = 2y - 10, \quad y(0) = 4$

$= 2(y-5)$

$y = 5 + ce^{2t}$

$y(0) = 5 + ce^0 = 5 + c \overset{\text{set}}{=} 4 \therefore c = -1$

$\boxed{y = 5 - e^{2t}}$

Note: I gave no partial credit on this problem.

(e) $\frac{dy}{dt} = ky, \quad y(0) = y_0$ $y = y_0 e^{kt}$

(f) $\frac{dy}{dt} = (y-2)\cos(ty), \quad y(0) = 2$

$y = 2$ (the constant solution)

Check: $0 \overset{?}{=} (2-2)\cos(t \cdot 2)$

$0 = 0$ ✓

6. Find the particular solution to

$$\frac{dy}{dx} = -\frac{3x^2+y}{x+2y}, \quad y(0)=1$$

ODE is not separable and not linear.
Let's hope it is exact!
Cross multiply: $(x+2y)dy = -(3x^2+y)dx$

Put in differential form $\boxed{(3x^2+y)dx + (x+2y)dy = 0}$

Mixed Partial test for exactness: $\frac{\partial}{\partial y}(3x^2+y) \stackrel{?}{=} \frac{\partial}{\partial x}(x+2y)$

$1 = 1$ Exact

So the ODE is really $dF = 0$ in disguise: $F_x dx + F_y dy = 0$
and the general solution is $F(x,y) = C$.
The super-important 2 things we know about F:

$\boxed{\text{I. } F_x(x,y) = 3x^2+y \qquad \text{II } F_y(x,y) = x+2y}$ Write these down!

$\text{I} \Rightarrow F(x,y) = \int (3x^2+y)dx = x^3 + xy + g(y)$

$\text{II} \Rightarrow F(x,y) = \int (x+2y)dy = xy + y^2 + h(x)$

So $\boxed{F(x,y) = x^3+xy+y^2}$ is one choice for F. NOT a solution to the ODE!

Method B: Plug expression from I into II

$\frac{\partial}{\partial y}(x^3+xy+g(y)) = x+2y$, $x + g'(y) = x+2y$, so $g'(y) = 2y$

$g(y) = y^2 + C$

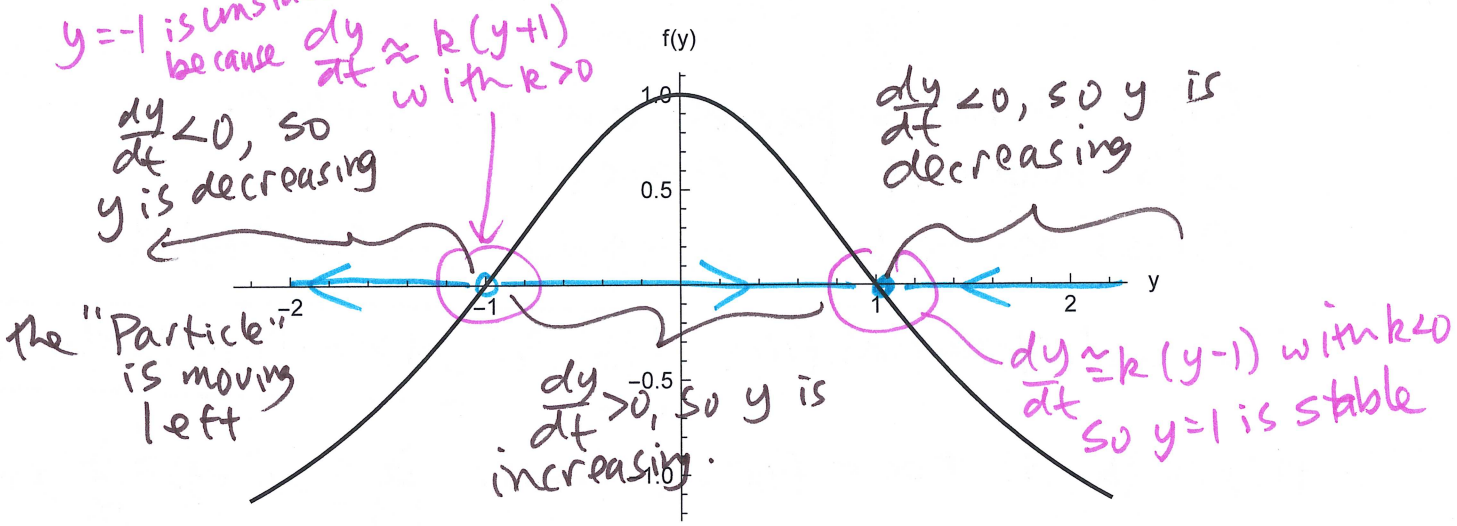
The general solution is $\boxed{x^3+xy+y^2 = C}$

$y(0)=1$ implies $0^3 + 0\cdot 1 + 1^3 = C$, so $C = 1$

the particular solution is $\boxed{x^3+xy+y^2 = 1}$

Red version

7. Consider the ODE $\frac{dy}{dt} = f(y)$, where the function f is plotted below:



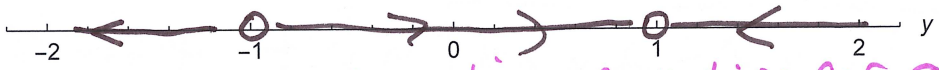
(a) Write down the equilibrium solutions, also known as constant solutions.

Solve $f(y) = 0$ for y: $\boxed{y=-1}$ and $\boxed{y=1}$

That is, $y(t) = -1$ for all t

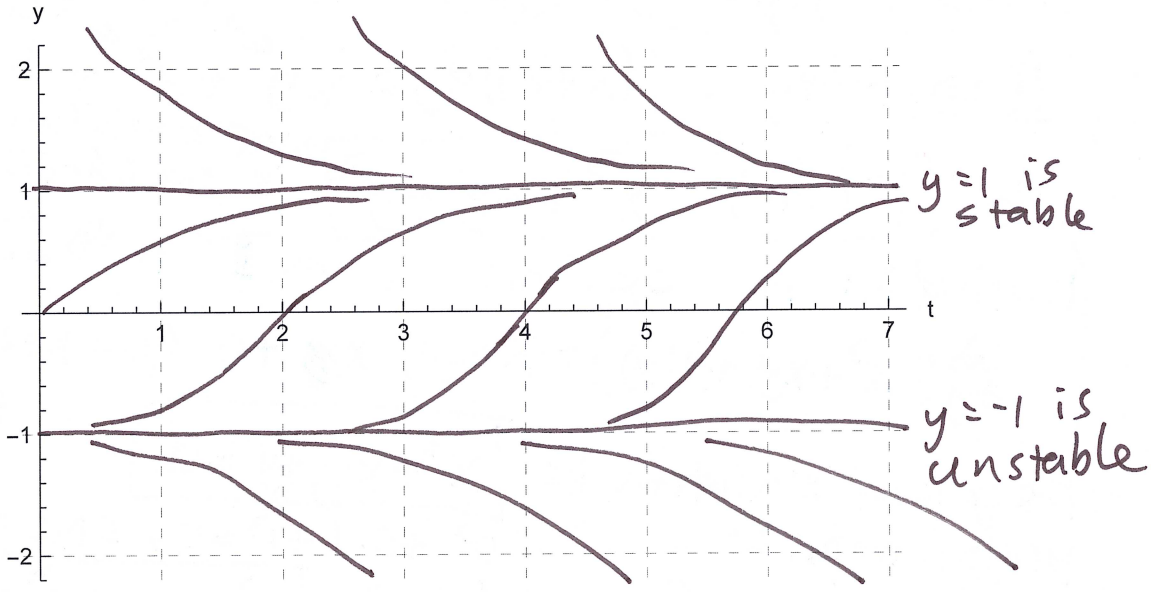
Note: "1" is not a solution

(b) Sketch the phase portrait of the ODE on the y-axis shown below.



This shows direction of motion for a particle moving with velocity $f(y)$ at position y.

(c) Sketch several solutions to the ODE in the t-y plane below. Be sure to draw the equilibrium solutions, and indicate if they are a stable equilibrium or an unstable equilibrium. Furthermore, draw the slope of the solution curves accurately as they cross the t axis.



Red version