MAT 239, Differential Equations, Prof. Swift
Final Exam Formula sheet

Name: _____

You must sign and turn in this formula sheet with the exam.

The quadratic formula for the solutions to $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Know the special right triangles (30-60-90 degrees, and 45-45-90 degrees) and the unit circle.

First order ODEs:

The standard form for a first order linear ODE is $y' + p(x)y = g(x)$. This can be solved using the integrating factor $\mu(x) = \exp(\int p(x)\,dx)$. A first order ODE in differential form, $P(x,y)dx + Q(x,y)dy = 0$, is exact if and only if $P_y(x,y) = Q_x(x,y)$. If the ODE is exact, the solutions are on level curves $F(x,y) = C$, with $F_x = P$ and $F_y = Q$.

The interval of existence of the IVP $y' = f(x,y)$, $y(x_0) = y_0$ is the largest interval containing x_0 on which the solution $y(x)$ is defined.

Euler's Method for $y' = f(x,y)$, $y(x_0) = y_0$ with step size h is
$x_{n+1} = x_n + h$, $y_{n+1} = y_n + h \cdot f(x_n, y_n)$.

Linear ODEs of order 2 or higher

A real root r of the characteristic equation corresponds to a solution $y = e^{rt}$ (or $y = e^{rx}$) of a Linear Homogeneous ODE with Constant Coefficients (LHODECC).

A complex conjugate pair of roots $a \pm ib$ of the characteristic equation corresponds to two solutions $y_1(t) = e^{at}\cos(bt)$ and $y_2(t) = e^{at}\sin(bt)$ of the LHODECC.

Repeated roots introduce factors of t to get linearly independent solutions of the LHODECC.

For linear nonhomogeneous ODEs, $y = y_h + y_p$. The Method of Undetermined Coefficients works for some Linear Nonhomogeneous ODEs with Constant Coefficients (LNODECCs).

Rule 1. The original form of y_p is the general solution of the lowest order LHODECC for which the general solution contains $g(t)$. Use A, B, C, etc. instead of the constants c_1, c_2, c_3, etc.
Rule 2. If necessary, use annihilators. The result is that some terms in y_p get multiplied by t^s (or x^s). No terms in the new $y_p(t)$ are also in $y_h(t)$.

$c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$, where $c_1 = R\cos(\delta)$, $\quad c_2 = R\sin(\delta)$

$my'' + \gamma y' + ky = F_0\cos(\omega t)$ with positive constants $m, \gamma, k, F_0, \omega$ has a particular solution $y_p(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$, where

$$A = \frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad B = \frac{F_0\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta = \arccos\left(\frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\right)$$

and $\omega_0 = \sqrt{\frac{k}{m}}$.

Formulas for Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1 \ (R = 1)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x \ (R = \infty)$$

You do not need to memorize the series for $\sin(x)$, $\cos(x)$, $\ln(1+x)$, etc.

$y(x) = \sum_{n=0}^{\infty} c_n x^n$ satisfies $y(0) = c_0$, $y'(0) = c_1$, and $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

Formulas for Systems of 1st Order Linear ODEs

The eigenvalues of A satisfy $\det(A - \lambda I) = 0$, and the associated eigenvectors satisfy $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$, which is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Eigenvalue hack: For the special case of $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, the eigenvalues satisfy $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \cdot \lambda_2 = ad - bc$.

These formula concern the solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where A is a 2×2 matrix with constant, real entries. Only in Case 1 do you need to find the eigenvectors of A. But you always need to find the eigenvalues of A. (I suggest you check the eigenvalues using the eigenvalue hack.)

Case 1. A has real, distinct eigenvalues, $\lambda_1 \neq \lambda_2$.

The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. For an IVP, the constants c_1 and c_2 are determined by the initial condition.

Case 2. A has complex eigenvalues $\lambda = a \pm ib$. To solve the IVP $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, first compute $\mathbf{x}_1 = (A - aI)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 \cos(bt) + \mathbf{x}_1 \tfrac{1}{b} \sin(bt)\right)$$

Case 3. A has repeated, real eigenvalues, $\lambda_1 = \lambda_2$. To solve the IVP $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, first compute $\mathbf{x}_1 = (A - \lambda_1 I)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{\lambda_1 t}\left(\mathbf{x}_0 + \mathbf{x}_1 t\right)$$