MAT 239, Differential Equations, Prof. Swift

Spring 2026 Final Exam:

Tuesday, May 5, in our usual classroom, for both sections:

Morning class (section 3) 7:30-9:30 am

Afternoon class (section 4) 12:30-2:30 pm

The test format will be similar to the previous exams.

The exam will begin and end promptly according to the scheduled time.

No notes are allowed. No calculators, phones, or watches are allowed. Put your phone and watch in your backpack.

A list of formulas will be printed on the exam. A pdf will be posted to the exams page on our website.

Possession of a calculator, phone, or watch during the exam, will be considered cheating. Looking at or copying another person's exam is also cheating.

There are no hard integrals if you do the problems correctly. Show your work! If you don't show your work, or have errors in the work, I may take off points even if you have the correct answer.

All problems have equal weight.

I will send email to the class when the final grades are uploaded to Canvas. I have to turn in the grades on Tuesday after finals week and I hope to post the grade summaries by the day before that. You can come to my office during the break or next semester to look at your graded final exam, but I need to keep it for my records.

Here is a sample Final Exam. The scanned solutions will be posted to our website. I suggest you try the exam *before* you look at the solutions.

As usual, some topics are left out of this sample exam due to time constraints. Your exam will most likely include questions about a different selection of topics. Anything covered on the three midterms, the four sample exams, or the in-class worksheets or quizzes might be on your exam.

1. Solve the ODE. (You do not need to show your work *on this problem.* Just write down the solution, if you know it.)

(a) $\dfrac{dy}{dt} = 3y$, $y(0) = 2$.

(b) $\dfrac{d^2y}{dt^2} = -9y$

(c) $\dfrac{dy}{dt} = 2(y-1)$

(d) $\dfrac{d^3y}{dt^3} = 0$

(e) $\dfrac{d^2y}{dt^2} + t^2\dfrac{dy}{dt} + y^2 = 0$, $y(0) = 0$, $y'(0) = 0$.

2. Find the general solution to $y' + \dfrac{2}{t}\,y = t$.

3. (a) Find an explicit solution to the IVP $\dfrac{dy}{dt} = y^3$, $y(0) = 1$.

(b) What is the interval on which the solution to part (a) is defined?

(c) Sketch the solution to part (a).

4. Match the first order ODE to the description. Put a letter in the blank. Hint: Use the differential form of the ODE to test exactness.

A: $y' = t^2 + y^2$, or $dy - (t^2 + y^2)dt = 0$

B: $y' = y \tan t$, or $dy - y \tan t \; dt = 0$

C: $y' = \dfrac{y^2}{\cos t}$, or $\cos t \; dy - y^2 dt = 0$

D: $y' = \dfrac{1 - y^2}{2yt}$, or $(2yt)dy - (1 - y^2)dt = 0$

E: $y' = e^{-t}(1 + y)$, or $dy - e^{-t}(1 + y)dt = 0$

F: $y' = \dfrac{-2yt}{y^2 + t^2}$, or $(y^2 + t^2)dy + 2yt \, dt = 0$

____ Linear homogeneous

____ Linear nonhomogeneous

____ Nonlinear and separable but not exact

____ Nonlinear and exact but not separable

____ Nonlinear and separable and exact

____ None of the above

Note: The following question was given at a test where graphing calculators were expected. You may not use a calculator this semester so you will not see a question like this. However, you might be asked to do a few steps of Euler's method by hand.

5. Use Euler's method with step size $h = 0.1$ to approximate $y(2)$ for the IVP $y' = x^2 + y$, $y(0) = 1$. Use your calculator. Write down the initialization step, and the command you repeated, as well as your answer rounded to 4 significant figures.

6. Find the solution to $(3x^2 + y)\,dx + (x + \cos(y))\,dy = 0$ with initial condition $y(1) = \pi$.

7. Find the form of the particular solution, but do not evaluate the undetermined coefficients, of the ODE

$$y''' + y' = te^t \sin(t) + 3t^2 + t - 1$$

(Note: There are 2 problem 7s.)

7. In this problem we will model pollution of a lake. Assume that the volume of the lake is 10^6 cubic meters. Initially the water in the lake is pure, but at time $t = 0$ a trans fat factory starts polluting the lake by pumping in effluent with a concentration of 2 kilograms per cubic meter of "substance F". The effluent is pumped in at a rate of 100 cubic meters per day. There is also a source of pure water entering the lake at a rate of 900 cubic meters per day. The pollutant in the lake is well mixed, and the mixed water drains out of the lake at the rate of 1000 cubic meters per day. (Thus, the volume of the lake stays at a constant 10^6 cubic meters.)

(a) Write down the IVP for $y(t)$, the kilograms of substance F in the lake after the factory has been open t days.

(b) Solve the IVP. (You may solve by inspection.)

(c) How many kilograms of substance F are in the lake in the limit $t \to \infty$?

8. Solve the IVP $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

9. This problem concerns a linear nonhomogeneous ODE $L[y] = g(t)$.

Suppose that:

$y_1(t)$ satisfies the ODE with the initial conditions $y(0) = 1$, $y'(0) = 0$,

$y_2(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 1$, and

$y_3(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 0$.

Find the solution to the ODE with the initial conditions $y(0) = 3$, $y'(0) = 4$.

(Your answer will be a linear combination of y_1, y_2, and y_3.)

10. Find the general solution to $x^2\,y'' - x\,y' - 3y = x^2$.

11. Find the recurrence relation for the series solution $y = \sum_{n=0}^{\infty} c_n x^n$ to

$$(1 + x^2)y'' + 4y = 0.$$

Do NOT solve the recurrence relation.

12. Suppose that the recurrence relation to an ODE is

$$c_{n+2} = \frac{-c_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, \cdots$$

(a) Find the first 3 nonzero terms in each of two linearly independent solutions to the ODE. (You may write your coefficients in terms of factorials.)

(b) (Extra credit) Sum each series to find 2 linearly independent solutions to the ODE.

13. Find the general solution to $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}$.

14. Find the general solution to $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \mathbf{x}$.

Hint: Find two linearly independent solutions by using two initial conditions, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The general solution is a linear combination of these two solutions.