MAT 239, Differential Equations, Prof. Swift
Spring 2007 Final Exam: Wednesday, May 9, 7:30 am - 9:30

Name: key

Four pages of notes with writing on both sides are allowed. A graphing calculator is expected. There are no difficult calculations or hard integrals if you use the correct approach. If you have a question, ask me. **Show your work!**

1. Solve the ODE. (You do not need to show your work *on this problem.* Just write down the solution, if you know it.)

(a) $\frac{dy}{dt} = 3y,\ y(0) = 2.$ $y = 2e^{3t}$

(b) $\frac{d^2y}{dt^2} = -9y$ $y = c_1\cos(3t) + c_2\sin(3t)$

(c) $\frac{dy}{dt} = 2(y-1)$ $y = 1 + ce^{2t}$ or $y = 1 + (y_0 - 1)e^{2t}$

(d) $\frac{d^3y}{dt^3} = 0$ $y = c_1 + c_2t + c_3t^2$

(e) $\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} + y^2 = 0,\ y(0) = 0,\ y'(0) = 0.$ $y = 0$

5 (2 pts)

2. Find the general solution to $y' + \frac{2}{t}y = t$.

$p(t) = \frac{2}{t}$

$\mu(t) = \exp\int \frac{2}{t}\,dt = \exp(2\ln t)$

$= \exp(\ln t^2) = t^2$

so $t^2y' + 2ty = t^3$

$(t^2y)' = t^3$

$t^2y = \frac{t^4}{4} + C$

$$y = \frac{t^2}{4} + \frac{c}{t^2}$$

3. (a) Find an explicit solution to the IVP $\frac{dy}{dt} = y^3$, $y(0) = 1$.
(b) What is the interval on which the solution to part (a) is defined?
(c) Sketch to solution to part (a).

$$\frac{dy}{y^3} = dt$$

$$\int y^{-3}\,dy = \int dt$$

$$\frac{y^{-2}}{-2} = t + C$$

Find C now:

$$\frac{1^{-2}}{-2} = 0 + C$$

$$-\frac{1}{2} = C$$

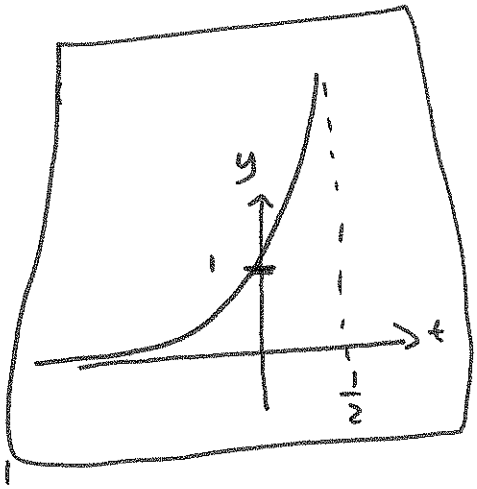
So $$\frac{y^{-2}}{-2} = t - \frac{1}{2}$$

$$y^{-2} = -2t + 1$$

$$\frac{1}{y^2} = 1 - 2t$$

$$y^2 = \frac{1}{1-2t}$$

$$\boxed{y = \frac{1}{\sqrt{1-2t}}}$$



(b) soln exists for $1-2t > 0$ or $2t < 1$

Solution defined for $\boxed{-\infty < t < \frac{1}{2}}$

or $\boxed{(-\infty, \frac{1}{2})}$

4. Match the first order ODE to the description. Put a letter in the blank. Hint: Use the differential form of the ODE to test exactness.

A: $y' = t^2 + y^2$, or $dy - (t^2 + y^2)dt = 0$

NL $\quad 0 \stackrel{?}{=} -2y$ NO.

B: $y' = y \tan t$, or $dy - y \tan t \; dt = 0$

C: $y' = \dfrac{y^2}{\cos t}$, or $\cos t \; dy - y^2 dt = 0$

NL, Sep. $\quad -\sin t \stackrel{?}{=} -2y$ No.

D: $y' = \dfrac{1 - y^2}{2yt}$, or $(2yt)dy - (1 - y^2)dt = 0$

$= \frac{1-y^2}{2yt} \cdot \frac{\frac{1}{t}}{\frac{1}{t}}$ Sep. $\quad 2y \stackrel{?}{=} 2y$ yes. exact

E: $y' = e^{-t}(1 + y)$, or $dy - e^{-t}(1 + y)dt = 0$

F: $y' = \dfrac{-2yt}{y^2 + t^2}$, or $(y^2 + t^2)dy + 2yt \, dt = 0$

NL, not sep $\quad 2t \stackrel{?}{=} 2t$ yes: exact.

B Linear homogeneous
E Linear nonhomogeneous
C Nonlinear and separable but not exact
F Nonlinear and exact but not separable
D Nonlinear and separable and exact
A None of the above

5. Use Euler's method with step size $h = 0.1$ to approximate $y(2)$ for the IVP $y' = x^2 + y$, $y(0) = 1$. Use your calculator. Write down the initialization step, and the command you repeated, as well as your answer rounded to 4 significant figures.

Initialization: $0 \to X : 1 \to Y$

Repeat $\quad Y + 0.1 * (X^2 + Y) \to Y : X + 0.1 \to X : \{X, Y\}$

$y(2) \approx 10.76$

6. Find the solution to $(3x^2 + y)\,dx + (x + \cos(y))\,dy = 0$ with initial condition $y(1) = \pi$.

$\frac{\partial}{\partial y}(3x^2+y) = 1 \quad \frac{\partial}{\partial x}(x+\cos(y)) = 1$

exact! Soln is $F(x,y) = C$

$\frac{\partial F}{\partial x} = 3x^2+y \quad$ so $F(x,y) = x^3 + xy + g(y)$

$\frac{\partial F}{\partial y} = x + \cos(y) \quad$ so $F(x,y) = xy + \sin(y) + h(x)$

So $F(x,y) = x^3 + xy + \sin(y)$

Now, find C

$F(1,\pi) = 1^3 + 1\cdot\pi + \sin(\pi) = 1+\pi$

So the solution is

$$\boxed{x^3 + xy + \sin(y) = 1+\pi}$$

7. Find the form of the particular solution, but do not evaluate the undetermined coefficients, of the ODE

$$y''' + y' = te^t\sin(t) + 3t^2 + t - 1$$

$k^3 + k = 0$

$(k^2+1)k = 0 \quad y_h = C_1 + C_2\cos(t) + C_3\sin(t)$

$Y = \ldots$

$$\boxed{Y = (At+B)e^t\sin(t) + (Ct+D)e^t\cos(t) + t(Et^2+Ft+G)}$$

7. In this problem we will model pollution of a lake. Assume that the volume of the lake is 10^6 cubic meters. Initially the water in the lake is pure, but at time $t = 0$ a trans fat factory starts polluting the lake by pumping in effluent with a concentration of 2 kilograms per cubic meter of "substance F". The effluent is pumped in at a rate of 100 cubic meters per day. There is also a source of pure water entering the lake at a rate of 900 cubic meters per day. The pollutant in the lake is well mixed, and the mixed water drains out of the lake at the rate of 1000 cubic meters per day. (Thus, the volume of the lake stays at a constant 10^6 cubic meters.)

(a) Write down the IVP for $y(t)$, the kilograms of substance F in the lake after the factory has been open t days.
(b) Solve the IVP. (You may solve by inspection.)
(c) How many kilograms of substance F are in the lake in the limit $t \to \infty$?

$\frac{dy}{dt}$ = rate in − rate out $\left(\frac{kg}{day}\right)$

rate in = $2\frac{kg}{m^3} \cdot 100\frac{m^3}{day} = 200\frac{kg}{day}$

rate out = $\frac{y}{10^6}\frac{kg}{m^3} \cdot 10^3\frac{m^3}{day} = \frac{y}{10^3}\frac{kg}{day}$

$y(0)=0$

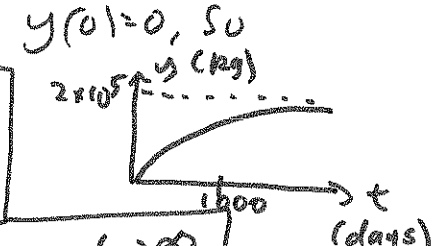
$$\frac{dy}{dt} = 200 - \frac{y}{10^3} = \frac{-y}{10^3} + \frac{10^3}{10^3} \cdot 200$$

$$= \frac{-1}{10^3}(y - 2\times10^5)$$

$$y = 2\times10^5 + C\,e^{-\frac{t}{10^3}}$$

$y(0)=0$, so

$$y = 2\times10^5\left(1 - e^{-\frac{t}{10^3}}\right)$$



2×10^5 kg of substance F as $t \to \infty$

8. Solve the IVP $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

$k^2 + 3k + 2 = 0$

$(k+1)(k+2) = 0$

$k_1 = -1 \quad k_2 = -2$

$y = c_1 e^{-t} + c_2 e^{-2t}$

$y' = -c_1 e^{-t} - 2c_2 e^{-2t}$

$y(0) = c_1 + c_2 = 0$

$y'(0) = -c_1 - 2c_2 = 1$

add $0 + -c_2 = 1$

$c_2 = -1$

~~$c_1 + c_2 = 0$~~

$y = -e^{-t} + e^{-2t}$

add $0 - c_2 = 1$

$c_2 = -1$

so $c_1 = 1$

$\boxed{y = e^{-t} - e^{-2t}}$

9. This problem concerns a linear nonhomogeneous ODE $L[y] = g(t)$.
Suppose that:
$y_1(t)$ satisfies the ODE with the initial conditions $y(0) = 1$, $y'(0) = 0$,
$y_2(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 1$, and
$y_3(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 0$.
Find the solution to the ODE with the initial conditions $y(0) = 3$, $y'(0) = 4$.
(Your answer will be a linear combination of y_1, y_2, and y_3.)

$L[c_1y_1 + c_2y_2 + c_3y_3] = (c_1 + c_2 + c_3)g(t)$

So $c_1 + c_2 + c_3 = 1$ to solve the ODE.

$c_1 = 3$ and $c_2 = 4$ to solve the IC. So $c_3 = -6$

$$\boxed{y = 3y_1 + 4y_2 - 6y_3}$$

10. Find the general solution to $x^2y'' - xy' - 3y = x^2$.

$y = y_h + Y$.

y_h solves $x^2y'' - xy' - 3y = 0$

try $y_h = x^r$

$r(r-1) - r - 3 = 0$

$r^2 - 2r - 3 = 0$

$(r+1)(r-3) = 0$

$r_1 = -1$ $r_2 = 3$

$y_h = c_1x^{-1} + c_2x^3$

Y has the form $Y = Ax^2$

$Y' = 2Ax$

$Y'' = 2A$

$x^2(2A) - x(2Ax) - 3Ax^2 = x^2$

$A = -\frac{1}{3}$ $Y = -\frac{1}{3}x^2$

$$\boxed{y = c_1x^{-1} + c_2x^3 - \frac{1}{3}x^2}$$

11. Find the recurrence relation for the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ to

$$(1+x^2)y'' + 4y = 0.$$

Do NOT solve the recurrence relation.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_n \cdot n\, x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n\, n(n-1)\, x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\, x^n$$

$$y'' + x^2 y'' + 4y = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\, x^n + \sum_{n=0}^{\infty} a_n\, n(n-1)\, x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n\, n(n-1) + 4a_n \right] x^n = 0$$

$$a_{n+2}(n+2)(n+1) + a_n(n^2 - n + 4) = 0$$

$$\boxed{a_{n+2} = \frac{-a_n(n^2 - n + 4)}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, \ldots}$$

12. Suppose that the recurrence relation to an ODE is

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, \cdots$$

(a) Find the first 3 nonzero terms in each of two linearly independent solutions to the ODE. (You may write your coefficients in terms of factorials.)
(b) (Extra credit) Sum each series to find 2 linearly independent solutions to the ODE.

$n=0$ $\quad a_2 = \frac{-a_0}{2\cdot 1} = \frac{-a_0}{2!}$

$n=1$ $\quad a_3 = \frac{-a_1}{3\cdot 2} = \frac{-a_1}{3!}$

$n=2$ $\quad a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2\cdot 1} = \frac{a_0}{4!}$

$n=3$ $\quad a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2} = \frac{a_1}{5!}$

$n=4$ not needed.

y_1 has $a_0=1$ and $a_1=0$

$$\boxed{y_1 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

y_2 has $a_0=0$ and $a_1=1$

$$\boxed{y_2 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}$$

(b) $\boxed{y_1 = \cos(x) \text{ and } y_2 = \sin(x)}$

Part (b) is worth 2 points.
You can get 100% of the class points without doing this part.

13. Find the general solution to $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x}$.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ You might know that the evals are $\lambda = 2 \pm 1$.

$\det(A - \lambda I) = 0$ is $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0$
$4 - 4\lambda + \lambda^2 - 1 = 0$
$\lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 1)(\lambda - 3) = 0$

OR

$(2-\lambda)^2 - 1 = 0$
$(2-\lambda)^2 = 1$
$2 - \lambda = \pm 1$
$\lambda = 2 \pm 1$.

$\lambda_1 = 1$, $\lambda_2 = 3$ (or the other way around).

$(A - \lambda_1 I)\vec{v}^{(1)} = \vec{0}$ is $(A - I)\vec{v}^{(1)} = \vec{0}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}^{(1)} = \vec{0}$$

one choice is $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$(A - \lambda_2 I)\vec{v}^{(2)} = \vec{0}$ is $(A - 3I)\vec{v}^{(2)} = \vec{0}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v}^{(2)} = \vec{0}$$

one choice is $\vec{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, the general solution is

$$\boxed{\vec{x} = C_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

14. Find the general solution to $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \vec{x}$.

$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$; $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-4) = 0$

$(1-\lambda)^2 = -4 \quad 1-\lambda = \pm 2i \quad \lambda = 1 \pm 2i = a + bi$ $\underline{a=1, b=2}$.

$A - aI = A - I = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Solution 1.

Let $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $(A - aI)\vec{x}_0 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \vec{x}_1$

$\vec{x} = e^t\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\frac{1}{2}\sin(2t)\right) = e^t \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$

Solution 2.

$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\vec{x} = e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\frac{1}{2}\sin(2t)\right) = e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

The general solution is the linear combination of these two solutions

$$\vec{x}(t) = C_1 e^t \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

This uses $\vec{x}(t) = e^{at}\left(\vec{x}_0\cos(bt) + \vec{x}_1\frac{1}{b}\sin(bt)\right)$, where $\vec{x}_1 = (A - aI)\vec{x}_0$ and eigenvalues of A are $a \pm ib$. Initial condition is $\vec{x}(0) = \vec{x}_0$