key to VersionA

1. (a) Find the explicit solution to the initial value problem $\frac{dy}{dx} = xy^2, \quad y(0) = 1.$

(b) Sketch the solution and indicate the interval on which the solution is defined. (This is also called the interval of existence, or the interval of validity.)

Separable.

$$\frac{dy}{y^2} = x\,dx$$

$$\int y^{-2}\,dy = \int x\,dx$$

$$\frac{y^{-1}}{-1} = \frac{x^2}{2} + C$$

$$\frac{-1}{y} = \frac{x^2}{2} + C$$

Plug in $y(0)=1$ and solve for C.

$$\frac{-1}{1} = \frac{0^2}{2} + C$$

$$C = -1$$

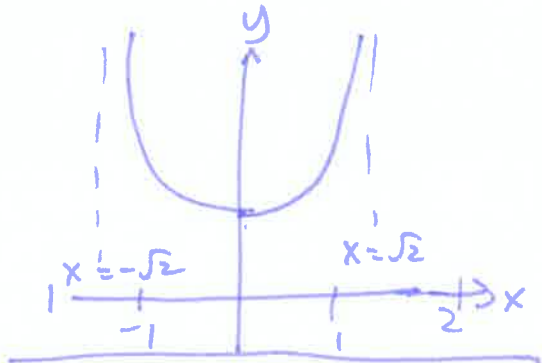
$$\frac{-1}{y} = \frac{x^2}{2} - 1$$

$$\frac{1}{y} = 1 - \frac{x^2}{2}$$

$$\boxed{y = \frac{1}{1 - \frac{x^2}{2}}} \quad \text{or} \quad \boxed{y = \frac{2}{2 - x^2}}$$

Vertical asymptotes at $x = \pm\sqrt{2}$

(b)



Solution is defined on $(-\sqrt{2}, \sqrt{2})$

Note: $f(x) = \frac{1}{1 - \frac{x^2}{2}}$ has the default domain $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$, but the solution to the IVP is defined on the interval that contains 0, namely $(-\sqrt{2}, \sqrt{2})$



Not part of solution!

2. (a) Find the solution to the IVP.

$$\frac{dy}{dt} + \frac{y}{t} = 4t^2, \quad y(1) = 0$$

(b) Sketch the solution and indicate the interval on which the solution is defined.

Linear, already in standard form

$p(t) = \frac{1}{t}$, $g(t) = 4t^2$

$\mu(t) = \exp\left(\int p(t)\,dt\right) = \exp\left(\int \frac{1}{t}\,dt\right)$

$= \exp(\ln(t))$

$= t$

$t\frac{dy}{dt} + y = 4t^3$

$\frac{d}{dt}(ty) = 4t^3$

$ty = \int 4t^3\,dt = t^4 + C$

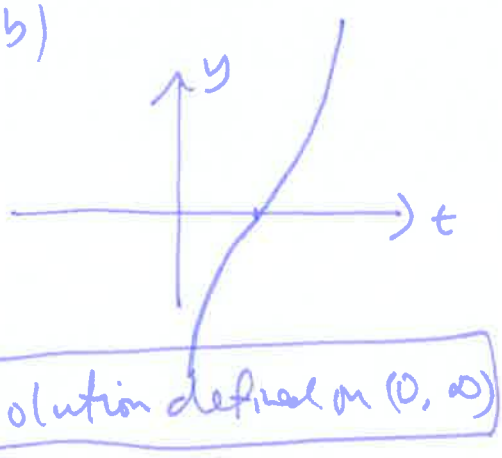
$ty = t^4 + C$

$y = t^3 + \frac{C}{t}$

$y(1) = 0$ $\quad 0 = 1^3 + \frac{C}{1}$ $\quad \therefore C = -1$

$\boxed{y = t^3 - \frac{1}{t}}$ solution to the IVP

(b)



solution defined on $(0, \infty)$

Note: $(-\infty, 0) \cup (0, \infty)$ is NOT an interval. It is the union of 2 intervals.

3. Find an implicit solution to the IVP

Not linear, not separable

$$(\sin(y) + x^2)dx + (x\cos(y) + 1)dy = 0, \quad y(0) = 1$$

$\frac{\partial}{\partial y}(\sin(y) + x^2) \stackrel{?}{=} \frac{\partial}{\partial x}(x\cos(y) + 1)$

$\cos(y) \stackrel{\checkmark}{=} \cos(y)$ $\therefore$ Exact.

ODE is exact, so there is a function $F(x,y)$ such that

$F_x(x,y) = \sin(y) + x^2$ $\therefore$ $F(x,y) = x\sin(y) + \frac{x^3}{3} + g(y)$

$F_y(x,y) = x\cos(y) + 1$ $\therefore$ $F(x,y) = x\sin(y) + y + h(x)$

So $F(x,y) = x\sin(y) + \frac{x^3}{3} + y + \cancel{c}^{\,0}$

general solution

$$x\sin(y) + \frac{x^3}{3} + y = C$$

Plug in $y(0) = 1$

$$0\sin(1) + \frac{0^3}{3} + 1 = C \quad \therefore C = 1$$

$$\boxed{x\sin(y) + \frac{x^3}{3} + y = 1}$$ is solution to the IVP

4. Suppose that you borrow \$10,000 for a car, at 5% interest. What should your monthly payment be so that you pay back the loan in 5 years?

Hint: Let $y(t)$ be the amount, in dollars, that you owe the bank after t years, and let P be your monthly payment. Assuming the payments are made continuously, y satisfies the IVP.

$$\frac{dy}{dt} = 0.05y - 12P, \qquad y(0) = 10000$$

Hint: $0.05 = \frac{1}{20}$.

$$\frac{dy}{dt} = \frac{1}{20}y - 12P = \frac{1}{20}\left(y - (12\cdot 20)P\right)$$

$$= \frac{1}{20}(y - 240P)$$

Solution (by inspection or other method)

$$y(t) = 240P + (10{,}000 - 240P)\,e^{\frac{1}{20}t}$$

What should P be so $y(5) = 0$?

$$0 = 240P + (10{,}000 - 240P)\,e^{\frac{1}{20}\cdot 5}$$

$$0 = 240P + (10{,}000 - 240P)\,e^{\frac{1}{4}}$$

$$0 = 240P(1 - e^{\frac{1}{4}}) + 10{,}000\,e^{\frac{1}{4}}$$

$$240P(1 - e^{\frac{1}{4}}) = -10{,}000\,e^{\frac{1}{4}}$$

$$P = \frac{-10{,}000\,e^{\frac{1}{4}}}{240(1 - e^{\frac{1}{4}})} = 188.37$$

Payment should be \$188.37 per month.

5. Put a "Y" or "N" in the blank, indicating if the ODE has the indicated property. Use the differential form to test exactness.

Y Separable Y Linear N Exact $\frac{dy}{dx} = \frac{y}{e^x}$ or $ydx - e^x dy = 0$.

$\frac{\partial}{\partial y}(y) \stackrel{?}{=} \frac{\partial}{\partial x}(-e^x)$

$1 \neq -e^x$ ∴ NOT exact

N Separable Y Linear N Exact $\frac{dy}{dx} = -y + x^2$ or $(y - x^2)dx + dy = 0$.

$\frac{\partial}{\partial y}(y - x^2) \stackrel{?}{=} \frac{\partial}{\partial x}(1)$

$1 \neq 0$ ∴ NOT exact

N Separable N Linear Y Exact $\frac{dy}{dx} = \frac{x + e^y}{1 - xe^y}$ or $(x + e^y)dx + (xe^y - 1)dy = 0$.

$\frac{\partial}{\partial y}(e^y) \stackrel{?}{=} \frac{\partial}{\partial x}(xe^y)$

$e^y = e^y$ ∴ exact

6. Compute two steps of Euler's method for the IVP $\frac{dy}{dx} = 2xy$, $y(0) = 1$ with $h = \frac{1}{2}$. That is, compute y_1 and y_2, that approximate $y(.5)$ and $y(1)$.

n	x_n	y_n	
0	0	1	
1	.5	1	$= 1 + \frac{1}{2}(2 \cdot 0 \cdot 1) = 1$
2	1	1.5	$= 1 + \frac{1}{2}(2 \cdot \frac{1}{2} \cdot 1) = 1 + \frac{1}{2}$

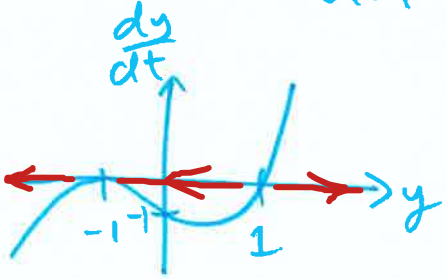
so $y_1 = 1$, $y_2 = 1.5$

7. Find the equilibrium solutions, and indicate if they are stable, unstable, or semistable, for the ODE

$$\frac{dy}{dt} = (y+1)^2(y-1) = y^3 + ?y^2 + ?y - 1$$

do you know this trick?

Constant solutions: $y = -1$, $y = 1$



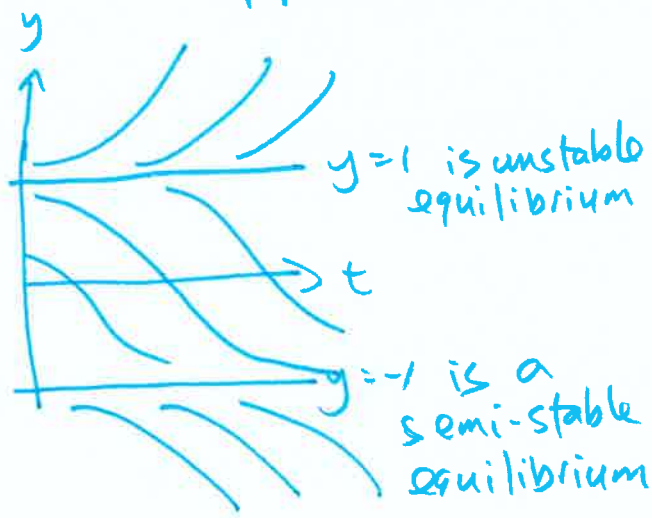


Phase Portrait

$\frac{dy}{dt} > 0$ y increasing

$\frac{dy}{dt} < 0$ y is decreasing

$\frac{dy}{dt} < 0$ y decreasing



$y = 1$ is unstable equilibrium

$y = -1$ is a semi-stable equilibrium

8. Solve the initial value problem by inspection and indicate the solution by putting a letter in the blank. You do not have to show your work.

A. $\frac{dy}{dt} = -2y, \quad y(0) = 1$ $y = 1e^{-2t}$ ___D___ $y = 0$

B. $\frac{dy}{dt} = 2(y-1), \quad y(0) = 1$ $y = 1 + Ce^{2t}$, $C = 0 \therefore y = 1$ ___C___ $y = 2e^{-t}$

C. $\frac{dy}{dt} = -y, \quad y(0) = 2$ $y = 2e^{-t}$ ___F___ $y = 2t + 1$

D. $\frac{dy}{dt} = -2y, \quad y(0) = 0$ $y = 0e^{-2t} = 0$ ___A___ $y = e^{-2t}$

E. $\frac{dy}{dt} = -2y + 4, \quad y(0) = 0$ $= -2(y-2)$ $y = 2 + Ce^{-2t}$, $C = -2$, $y = 2 - 2e^{-2t}$ ___E___ $y = 2 - 2e^{-2t}$

F. $\frac{dy}{dt} = 2, \quad y(0) = 1$ ___B___ $y = 1$

$y = 2t + C$

$y(0) = 0 + C = 1$

$C = 1$

$y = 2t + 1$

9. Match the ODE to the slope field. Put a letter in the box. Also, sketch the solution for each ODE with the initial condition $y(0) = 0$.

A. $y' = 1 - x^2$ B. $y' = 1 - y^2$ C $y' = y - x$
D. $y' = x^2 - 1$ E. $y' = y^2 - 1$ F. $y' = x - y$

