MAT 239, Differential Equations, Prof. Swift

Exam 2, on WeBWorK sets 9-13
will be Tuesday, March 31, 2026 (Week 11, day 1)

I suggest you take this sample exam in a single 75 minute sitting, following the rules of the exam (no notes or devices), before looking at the solutions.

The groundrules and formulas printed below will be printed on your exam.

Groundrules
The exam will begin at end promptly, in recognition that many students have classes at South campus before and/or after our class.

No notes are allowed. No calculators are allowed. No earbuds or earphones, or phones, or watches, or other electronics are allowed. They should be stored in your backpack.

Show your work! If you don't show your work, or you have errors in the work, I may take off points even if you have the correct answer. (Some problems indicate that you do not need to show your work.)

Formulas (updated March 27, 2026.)
A real root r of the characteristic equation corresponds to a solution $y = e^{rt}$ (or $y = e^{rx}$) of a Linear Homogeneous ODE with Constant Coefficients (LHODECC).

A complex conjugate pair of roots $a \pm ib$ of the characteristic equation corresponds to two solutions $y_1(t) = e^{at}\cos(bt)$ and $y_2(t) = e^{at}\sin(bt)$ of the LHODECC.

Repeated roots introduce factors of t to get linearly independent solutions of the LHODECC.

For linear nonhomogeneous ODEs, $y = y_h + y_p$. The Method of Undetermined Coefficients works for some Linear Nonhomogeneous ODEs with Constant Coefficients (LNODECCs).

Rule 1. The original form of y_p is the general solution of the lowest order LHODECC for which the general solution contains $g(t)$. Use A, B, C, etc. instead of the constants c_1, c_2, c_3, etc.

Rule 2. If necessary, use annihilators. The result is that some terms in y_p get multiplied by t^s (or x^s). No terms in the new $y_p(t)$ are also in $y_h(t)$.

$c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$, where $c_1 = R\cos(\delta)$, $\quad c_2 = R\sin(\delta)$

$my'' + \gamma y' + ky = F_0\cos(\omega t)$ with positive constants $m, \gamma, k, F_0, \omega$ has a particular solution $y_p(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$, where

$$A = \frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad B = \frac{F_0\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta = \arccos\left(\frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\right)$$

and $\omega_0 = \sqrt{\frac{k}{m}}$.

1. (a) Solve the Initial Value Problem (IVP) for $y(t)$. (You do not need to show your work on this part.)

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -2$$

(b) Write the solution in the form $y = R\cos(\omega_0 t - \delta)$. (Give the exact values of R and δ.)

2. Find the general solution to $y'' - 3y' + 2y = e^{-t}$.

3. (You do not need to show your work on this problem.) For this entire problem, $L[y]$ is the linear operator defined by

$$L[y] = e^{t^2}y'' + \arctan(t)y' + \sin(t)y.$$

Suppose $y_1(t)$ satisfies the Initial Value Problem (IVP)

$$L[y] = \ln(1 + t^2), \ y(0) = 1, \ y'(0) = 0,$$

and $y_2(t)$ satisfies the IVP

$$L[y] = \ln(1 + t^2), \ y(0) = 0, \ y'(0) = 1,$$

and $y_3(t)$ satisfies the IVP

$$L[y] = \ln(1 + t^2), \ y(0) = 0, \ y'(0) = 0.$$

Note: You may use "$L[y]$" in your answers.

(a) What IVP does $y = y_1(t) - y_2(t)$ satisfy?

(b) What IVP does $y = y_1(t) - y_3(t)$ satisfy?

(c) What IVP does $y = y_2(t) - y_3(t)$ satisfy?

(d) Using y_1, y_2, and y_3 as given functions, find the solution to the IVP

$$L[y] = 0, \ y(0) = \alpha, \ y'(0) = \beta,$$

where α and β are arbitrary constants.

4. Suppose that the characteristic equation for a linear homogeneous ODE with constant coefficients has the roots listed. Write down the general solution, $y(t)$, in each case: (You do not need to show your work on this problem.)

(a) $r = -1, \ 2 \pm i$ (b) $r = 1, 1, 1$ (c) $r = 1 \pm 3i, \ 1 \pm 3i$

5. Solve the IVP $y'' + 4y' + 3y = 0$, $y(0) = 0$, $y'(0) = 1$.

6. A particle is subject to a constant gravitational force, and friction proportional to the velocity. Solve for the position $y(t)$, which satisfies

$$y'' + 4y' = 32, \qquad y(0) = 0, \quad y'(0) = 0.$$

7. (a) Find the general solution to $x^2y'' - 2y = 0$.

(b) Find a particular solution to $x^2y'' - 2y = 8x^3$.

(c) Find the general solution to $x^2y'' - 2y = 8x^3$.

8. Match the ODEs to the general solution. (Put one letter in each blank.) In this problem, $D = \frac{d}{dt}$.

A. $y'' + 16y = 0$ $\qquad$ $y(t) = c_1\cosh(4t) + c_2\sinh(4t)$ _____

B. $y'' - 16y = 0$ $\qquad$ $y(t) = e^{3t}(c_1 + c_2t)$ _____

C. $(D-3)^2y = 0$ $\qquad$ $y(t) = c_1 + c_2t$ _____

D. $y'' = 0$ $\qquad$ $y(t) = e^{2t}(c_1 + c_2t + c_3t^2)$ _____

E. $y''' + 4y = 0$ $\qquad$ $y(t) = c_1\cos(4t) + c_2\sin(4t)$ _____

F. $y''' - 8y = 0$ $\qquad$ $y(t) = c_1 + c_2t + c_3t^2$ _____

G. $(D-2)^3y = 0$

H. $y''' = 0$

9. Match the ODE to the form of a particular solution. (Put each letter in one blank. More than one letter can go in a given blank.) In this problem, $D = \frac{d}{dt}$.

A. $(D-2)^2y = 3\sin(2t)$ $\qquad$ $y_p(t) = A\cos(2t)$ _____

B. $y'' + 4y = 3\sin(2t)$ $\qquad$ $y_p(t) = A\sin(2t)$ _____

C. $(D-2)^2y = 5te^{2t}$ $\qquad$ $y_p(t) = A\cos(2t) + B\sin(2t)$ _____

D. $(D-1)(D-2)y = 4\cos(2t)$ $\qquad$ $y_p(t) = At\cos(2t) + Bt\sin(2t)$ _____

E. $y'' - 4y = e^{2t}$ $\qquad$ $y_p(t) = At$ _____

F. $y'' + 4y = t$ $\qquad$ $y_p(t) = At + B$ _____

$y_p(t) = Ae^{2t}$ _____

$y_p(t) = Ate^{2t}$ _____

$y_p(t) = (At + B)e^{2t}$ _____

$y_p(t) = (At^3 + Bt^2)e^{2t}$ _____