MT2 Version 1

1. (a) Solve the Initial Value Problem (IVP) for $y(t)$. (You do not need to show your work on this part.)

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -2$$

$k^2+4=0$, $k^2=-4$, $k=\pm 2i$, or solve by inspection

$\boxed{y = \cos(2t) - \sin(2t)}$ or $y = c_1\cos(2t) + c_2\sin(2t)$

$y' = -2c_1\sin(2t) + 2c_2\cos(2t)$

$c_1 = 1$

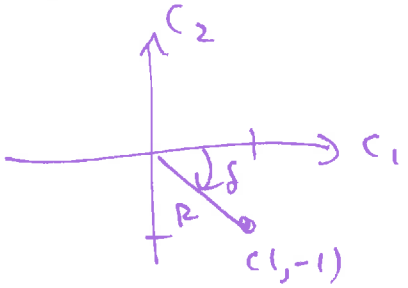
$2c_2 = -2 \therefore c_2 = -1$

(b) Write the solution in the form $y = R\cos(\omega_0 t - \delta)$. (Give the exact value, not a calculator approximation, of R and δ.)

$c_1 = 1$, $c_2 = -1$

$c_1 = R\cos(\delta)$

$c_2 = R\sin\delta$



$R = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\tan\delta = \frac{c_2}{c_1} = -1$

and δ in quadrant IV

$\delta = -\frac{\pi}{4}$

so $\boxed{y = \sqrt{2}\cos\left(2t + \frac{\pi}{4}\right)}$

2. Find the general solution to

$$y'' - 3y' + 2y = e^{-t}$$

$k^2 - 3k + 2 = 0$

$(k-1)(k-2) = 0$

$k_1 = 1, k_2 = 2$ so $\boxed{y_h = c_1e^t + c_2e^{2t}}$

Undet. coeff.

$y_p = Ae^{-t}$

$y_p' = -Ae^{-t}$

$y_p'' = Ae^{-t}$

$Ae^{-t} - 3(-Ae^{-t}) + 2(Ae^{-t}) = e^{-t}$

$(1+3+2)Ae^{-t} = e^{-t}$

$6A = 1 \therefore A = \frac{1}{6}$

$\boxed{y_p = \frac{1}{6}e^{-t}}$

General solution is $(y = y_h + y_p)$

$\boxed{y = c_1e^t + c_2e^{2t} + \frac{1}{6}e^{-t}}$

3. (You do not need to show your work on this problem.) For this entire problem, $L[y]$ is the linear operator defined by

$$L[y] = e^{t^2}y'' + \arctan(t)y' + \sin(t)y.$$

Suppose $y_1(t)$ satisfies the Initial Value Problem (IVP)

$$L[y] = \ln(1+t^2),\ y(0) = 1,\ y'(0) = 0,$$

and $y_2(t)$ satisfies the IVP

$$L[y] = \ln(1+t^2),\ y(0) = 0,\ y'(0) = 1,$$

and $y_3(t)$ satisfies the IVP

$$L[y] = \ln(1+t^2),\ y(0) = 0,\ y'(0) = 0.$$

Note: You may use "$L[y]$" in your answers.

(a) What IVP does $y = y_1(t) - y_2(t)$ satisfy?

$L[y] = 0, \quad y(0) = 1, \quad y'(0) = -1$

(b) What IVP does $y = y_1(t) - y_3(t)$ satisfy?

$L[y] = 0 \quad y(0) = 1, \quad y'(0) = 0$

(c) What IVP does $y = y_2(t) - y_3(t)$ satisfy?

$L[y] = 0 \quad y(0) = 0 \quad y'(0) = 1$

(d) Using y_1, y_2, and y_3 as given functions, find the solution to the IVP

2pts

$$L[y] = 0,\ y(0) = \alpha,\ y'(0) = \beta,$$

where α and β are arbitrary constants.

mult. answer from (b) by α and answer from (c) by β.

$$y = \alpha(y_1(t) - y_3(t)) + \beta(y_2(t) - y_3(t))$$

4. Suppose that the characteristic equation for a linear homogeneous ODE with constant coefficients has the roots listed. Write down the general solution, $y(t)$, in each case: (You do not need to show your work on this problem.)

(a) $k = -1,\ 2 \pm i$ $y(t) = C_1 e^{-t} + C_2 e^{2t}\cos(t) + C_3 e^{2t}\sin(t)$

(b) $k = 1, 1, 1$ $y(t) = C_1 e^{t} + C_2 t e^{t} + C_3 t^2 e^{t} = e^{t}(C_1 + C_2 t + C_3 t^2)$

(c) $k = 1 \pm 3i,\ 1 \pm 3i$ $y(t) = (C_1 + C_2 t) e^{t}\cos(3t) + (C_3 + C_4 t) e^{t}\sin(3t)$

5. Solve the IVP $y'' + 4y' + 3y = 0$, $y(0) = 0$, $y'(0) = 1$.

$k^2 + 4k + 3 = 0$

$(k+1)(k+3) = 0$

$k_1 = -1,\ k_2 = -3$

$y = C_1 e^{-t} + C_2 e^{-3t}$ is general solution

$y'(t) = -C_1 e^{-t} - 3C_2 e^{-3t}$

$y(0) = C_1 + C_2 \overset{\text{set}}{=} 0$

$y'(0) = -C_1 - 3C_2 \overset{\text{set}}{=} 1$

add: $0 - 2C_2 = 1$ $\therefore C_2 = -\frac{1}{2}$

1st eqn $C_1 = -C_2 = \frac{1}{2}$

$$\boxed{y = \tfrac{1}{2} e^{-t} - \tfrac{1}{2} e^{-3t}}$$

6. (a) A particle is subject to a constant gravitational force, and friction proportional to the velocity. Solve for the position $y(t)$, which satisfies

$$y'' + 4y' = 32, \qquad y(0) = 0, \quad y'(0) = 0.$$

y_h satisfies $y'' + 4y' = 0$

$k^2 + 4k = 0$

$k(k+4) = 0$

$k = 0, -4$ $\qquad y_h = C_1 + C_2 e^{-4t}$

$y_p = At$ (Rule II applies)

$y_p' = A$ $\qquad 0 + 4A = 32$

$y_p'' = 0$ $\qquad A = 8$ so $y_p = 8t$

gen soln: $y = C_1 + C_2 e^{-4t} + 8t$ $\qquad y(0) = C_1 + C_2 = 0$

(b) Solve the following initial value problem for $y(t)$, where m, γ, and g are constants.

$$my'' + \gamma y' = mg, \qquad y(0) = 0, \quad y'(0) = 0$$

$y' = -4C_2 e^{-4t} + 8$ $\qquad y'(0) = -4C_2 + 8 = 0$

So $C_1 + C_2 = 0$

$-4C_2 + 8 = 0$ or $\qquad 4C_2 = 8$, $C_2 = 2$

$C_1 = -2$

$$\boxed{y = -2 + 2e^{-4t} + 8t}$$

7. (a) Find the general solution to

$$x^2y'' - 2y = 0$$

try $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$x^2 r(r-1)x^{r-2} - 2x^r = 0$

$x^r(r(r-1) - 2) = 0$

$r^2 - r - 2 = 0$

$(r+1)(r-2) = 0$

$r = -1, 2$

So $y(x) = C_1x^{-1} + C_2x^2$

(b) Find a particular solution to

$$x^2y'' - 2y = 8x^3$$

try $y_p = Ax^3$, $y_p' = 3Ax^2$, $y_p'' = 6Ax$

$x^2(6Ax) - 2Ax^3 = 8x^3$

$6Ax^3 - 2Ax^3 = 8x^3$

$4Ax^3 = 8x^3$

$4A = 8$

$A = 2$

$y_p = 2x^3$

(c) Find the general solution to

$y = y_h + y_p$

$$x^2y'' - 2y = 8x^3$$

$y(x) = C_1x^{-1} + C_2x^2 + 2x^3$

8. Match the ODEs to the general solution. (Put one letter in each blank.) In this problem, $D = \frac{d}{dt}$.

A. $y'' + 16y = 0$ $r^2+16=0$ $r=\pm 4i$ $y(t) = c_1\cos(4t) + c_2\sin(4t)$ B

B. $y'' - 16y = 0$ $r^2-16=0$ $r=\pm 4$ $y(t) = e^{3t}(c_1 + c_2t)$ C

C. $(D-3)^2y = 0$ $r=3,3$ $y(t) = c_1 + c_2t$ D

D. $y'' = 0$ $r=0,0$ $y(t) = e^{2t}(c_1 + c_2t + c_3t^2)$ G

E. $y''' + 4y = 0$ $r^3+4=0$ (?) $y(t) = c_1\cos(4t) + c_2\sin(4t)$ A

F. $y''' - 8y = 0$ $r^3-8=0$ $r=2$, ?? $y(t) = c_1 + c_2t + c_3t^2$ H

G. $(D-2)^3y = 0$ $r=2,2,2$

H. $y''' = 0$ $r=0,0,0$

9. Match the ODE to the form of a particular solution. (Put each letter in one blank. More than one letter can go in a given blank.) In this problem, $D = \frac{d}{dt}$.

A. $(D-2)^2y = 3\sin(2t)$ $r=2,2$ $r'=\pm 2i$ $y_p(t) = A\cos(2t)$ ____

B. $y'' + 4y = 3\sin(2t)$ $r=\pm 2i$ $r'=\pm 2i$ $y_h = c_1\cos(2t)+c_2\sin(2t)$ $y_p(t) = A\sin(2t)$ ____

C. $(D-2)^2y = 5te^{2t}$ $r=2,2$ $r'=2,2$ $y_p(t) = A\cos(2t) + B\sin(2t)$ A, D

D. $(D-1)(D-2)y = 4\cos(2t)$ $r=1,2$ $r'=\pm 2i$ $y_p(t) = At\cos(2t) + Bt\sin(2t)$ B

$= t(A\cos(2t)+B\sin(2t))$

E. $y'' - 4y = e^{2t}$ $r=\pm 2$ $r'=2$ $y_h = c_1e^{2t}+c_2e^{-2t}$ $y_p(t) = At$ ____

F. $y'' + 4y = t$ $r=\pm 2i$ $r'=0,0$ $y_p(t) = At + B$ F

$y_p(t) = Ae^{2t}$ ____

$y_p(t) = Ate^{2t}$ E

$y_p(t) = (At + B)e^{2t}$ ____

$y_p(t) = (At^3 + Bt^2)e^{2t}$ C

$= t^2(At+B)e^{2t}$