MAT 239, Differential Equations, Prof. Jim Swift

Exam 3 on WeBWorK sets 14-17 will be in class Thursday, April 23, 2026 (Week 14)

The test format will be similar to the first two exams.

The exam will begin at end promptly, in recognition that many students have classes at South campus before and/or after our class.

No notes are allowed. No calculators are allowed. No earbuds or earphones, or phones, or watches, or other electronics are allowed. They should be stored in your backpack.

Show your work! If you don't show your work, or you have errors in the work, I may take off points even if you have the correct answer. (Some problems indicate that you do not need to show your work.)

The next page has an exam from a previous semester. I suggest you give yourself 75 un-interrupted minutes to try the exam before looking at the solutions.

To study for the exam, start with this practice exam, then look at the in-class worksheets, then look at the webwork. Here are suggested problems from each set.
Set 14: 3, 5. Set 15: 1, 2, 3, 4. Set 16: 3, 6, 11, 13. Set 17: 2, 3, 6, 8.

The following formulas will be given to you at the exam.

The quadratic formula for the solutions to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Formulas for Power Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1 \ (R = 1)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x \ (R = \infty)$$

You do not memorize the power series of other functions. I will give them to you if needed.

$y(x) = \sum_{n=0}^{\infty} c_n x^n$ satisfies $y(0) = c_0$, $y'(0) = c_1$, and $y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

More formulas on the next page.

Formulas for Systems of 1st Order Linear ODEs

The eigenvalues of A satisfy $\det(A - \lambda I) = 0$, and the associated eigenvectors satisfy $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$, which is equivalent to $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Eigenvalue hack: For the special case of $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, the eigenvalues satisfy $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \cdot \lambda_2 = ad - bc$.

These formula concern the solution to $\mathbf{x}' = A\mathbf{x}$, where A is a 2×2 matrix with constant, real entries.

Case 1. A has real, distinct eigenvalues, $\lambda_1 \neq \lambda_2$.
The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. The constants c_1 and c_2 are determined by the initial condition.

Case 2. A has complex eigenvalues $\lambda = a \pm ib$. To solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, first compute $\mathbf{x}_1 = (A - aI)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 \cos(bt) + \mathbf{x}_1 \tfrac{1}{b}\sin(bt)\right)$$

Case 3. A has repeated, real eigenvalues, $\lambda_1 = \lambda_2$. To solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, first compute $\mathbf{x}_1 = (A - \lambda_1 I)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{\lambda_1 t}\left(\mathbf{x}_0 + \mathbf{x}_1 t\right)$$

Sample Exam

1. Find the recurrence relation for the series solution $y = \sum_{n=0}^{\infty} c_n x^n$ to the ODE $y'' - xy' + 3y = 0$. *Do not* solve the recurrence relation.

2. Suppose that the series solution, $y = \sum_{n=0}^{\infty} c_n x^n$, to a second order linear ODE has the recurrence relation

$$c_{n+2} = \frac{c_n(n-3)}{(n+2)(n+1)} \ , \text{ for } n = 0, 1, 2, 3, \cdots$$

Find the first 3 nonzero terms in each of two linearly independent series solutions (unless the series terminates sooner). Indicate any *exact* solutions you find in this way.

3. (a) Find the characteristic equation of the second order ODE $y'' + 2y' + y = 0$. (This is a quadratic equation and was a main topic from Exam 2.) Then find the roots of the characteristic equation.

(b) Write the second order ODE from part (a) as its equivalent system of first order ODEs. Use v to represent the "velocity function." That is, $v(t) = y'(t)$. Write the system in matrix form as

$$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} — & — \\ — & — \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix}$$

(c) Find the eigenvalues of the 2×2 matrix you found in part (b). Point out anything interesting that you notice.

4. Find the general solution to $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\mathbf{x}$.

5. Find the solution to $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}\mathbf{x}$, with the inital condition $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

6. Find the eigenvalues of the matrix A, given that

$$A = \begin{bmatrix} 9 & 6 & 2 \\ -7 & -4 & -2 \\ -18 & -12 & -4 \end{bmatrix}$$

and its eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}.$$