V.1.

1. Find the recurrence relation for the series solution $y = \sum_{n=0}^{\infty} c_n x^n$ to the ODE $y'' - xy' + 3y = 0$. *Do not* solve the recurrence relation.

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n$$

$$y'' \quad - x y' \quad + 3y = 0$$

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n - x \sum_{n=1}^{\infty} n c_n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$'' \quad - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 3 c_n x^n = 0$$

$n=0$ gives same sum

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - n c_n + 3 c_n \right] x^n = 0$$

$[\cdots] = 0$ for $n = 0, 1, 2, \ldots$

this is part of the recurrence relation

$$(n+2)(n+1) c_{n+2} - (n-3) c_n = 0$$

$$c_{n+2} = \frac{(n-3) c_n}{(n+2)(n+1)}, \text{ for } n = 0, 1, 2, 3, \ldots$$

you may write the $n=0$ term separately

$$c_2 = -\frac{3}{2} c_0, \quad c_{n+2} = \frac{(n-3) c_n}{(n+2)(n+1)} \text{ for } n = 1, 2, 3, \ldots$$

V. 1

2. Suppose that the series solution, $y = \sum_{n=0}^{\infty} c_n x^n$, to a second order linear ODE has the recurrence relation

$$c_{n+2} = \frac{c_n(n-3)}{(n+2)(n+1)}, \text{ for } n = 0, 1, 2, 3, \cdots$$

Find the first 3 nonzero terms in each of two linearly independent series solutions (unless the series terminates sooner). Indicate any *exact* solutions you find in this way.

you can stop here with two and get $y = 1 + x - \frac{3}{2}x^2 + \cdots$ and $y = 1 + 2x - \frac{3}{2}x^2 + \cdots$ as 2 linearly indep. solns.

$n=0$: $C_2 = \frac{C_0(-3)}{2\cdot 1} = -\frac{3}{2}C_0$

$n=1$: $C_3 = \frac{C_1(-2)}{3\cdot 2} = -\frac{C_1}{3}$

$n=2$: $C_4 = \frac{C_2(-1)}{4\cdot 3} = -\frac{C_2}{12} = -\frac{\left(-\frac{3}{2}C_0\right)}{12} = \frac{3C_0}{24} = \frac{C_0}{8}$

$n=3$: $C_5 = \frac{C_3(0)}{5\cdot 4} = 0$ $\quad (C_7 = C_9 = C_{11} = \cdots = 0)$

Standard way to get 2 linearly independent solutions:

y_1 has $C_0 = 1, C_1 = 0$

$C_2 = -\frac{3}{2}$

$C_3 = 0$

$C_4 = \frac{1}{8}$ so $\boxed{y_1 = 1 - \frac{3}{2}x^2 + \frac{1}{8}x^4 + \cdots}$

y_2 has $C_0 = 0, C_1 = 1$

$C_2 = 0$

$C_3 = -\frac{1}{3}$

$C_4 = 0$

$C_5 = 0, C_6 = C_7 = C_8 = \cdots = 0$, so $\boxed{y_2 = x - \frac{1}{3}x^3 \text{ is an exact solution}}$

V. 1

3. (a) Find the characteristic equation of the second order ODE $y'' + 2y' + y = 0$. (This is a quadratic equation and was a main topic from Exam 2.) Then find the roots of the characteristic equation.

$y''+2y'+y=0$

$k^2+2k+1=0$ char. equation

$(k+1)^2=0$

$k=-1,-1.$ roots.

(b) Write the second order ODE from part (a) as its equivalent system of first order ODEs. Use v to represent the "velocity function." That is, $v(t) = y'(t)$. Write the system in matrix form as

$$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix}$$

$y' = v$

$v' = y'' = -2y' - y$

$= -2v - y$

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ -y-2v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix}$$ so $v' = -y - 2v$

(c) Find the eigenvalues of the 2×2 matrix you found in part (b). Point out anything interesting that you notice.

$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} \overset{\text{set}}{=} 0$

$(-\lambda)(-2-\lambda) - (1)(1) = 0$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\lambda = -1, -1$ are eigenvalues

These are also the roots found in (a)!

V.1

4. Find the general solution to $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{x}$. For full credit, show your work and do not use a calculator to find eigenvalues or eigenvectors.

$A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$ $\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 2 \\ 1 & -1-\lambda \end{bmatrix}\right) \overset{\text{set}}{=} 0$

$= (-\lambda)(-1-\lambda) - 2 = 0$

$= \lambda^2 + \lambda - 2 = 0$

$(\lambda - 1)(\lambda + 2) = 0$

$\lambda_1 = 1, \ \lambda_2 = -2$

$\lambda_1 = 1$

$A - \lambda_1 I = A - I = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$, so $\vec{V}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by inspecti

or $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{matrix} -a + 2b = 0 \\ a - 2b = 0 \end{matrix} \} \rightarrow a = 2b$

choose any $b \neq 0$

for example $b = 1$

to get $\vec{V}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda_2 = -2$

$A - \lambda_2 I = A + 2I = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$, so $\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ by inspectin is one choice.

using formula on front page, the general solution is

$$\vec{X} = C_1 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5. Find the solution to $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}\mathbf{x}$, with the inital condition $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

$A = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}$ $T = 3+3 = 6$ (Trace A)

$D = 3\cdot 3 - 4(-1)$ (Determinant of A.)
$= 9 + 4 = 13$

$\lambda_1 + \lambda_2 = 6$
$\lambda_1 \cdot \lambda_2 = 13$) No integer solutions. Go to char. eqn.

$\lambda^2 - 6\lambda + 13 = 0$: complete the square, or use quadratic formula.
$\lambda^2 - 6\lambda = -13$
$\lambda^2 - 6\lambda + 9 = -13 + 9$
$(\lambda - 3)^2 = -4$
$\lambda - 3 = \pm 2i$
$\boxed{\lambda = 3 \pm 2i}$

Case 2 in formula sheet, with $a = 3$, $b = 2$.

$\vec{X}_0 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $\vec{X}_1 = (A - 3I)\vec{X}_0 = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix}$

OK) $\vec{X}(t) = e^{3t}\left(\begin{bmatrix} 0 \\ 4 \end{bmatrix}\cos(2t) + \begin{bmatrix} 16 \\ 0 \end{bmatrix}\frac{1}{2}\sin(2t) \right)$

Simplifies $\boxed{\vec{X}(t) = e^{3t}\begin{bmatrix} 8\sin(2t) \\ 4\cos(2t) \end{bmatrix}}$ is the solution to the IVP

V. 3, V. 1

6. Find the eigenvalues of the matrix A, given that

$$A = \begin{bmatrix} 9 & 6 & 2 \\ -7 & -4 & -2 \\ -18 & -12 & -4 \end{bmatrix}$$

and its eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}.$$

$$A\vec{V}_1 = \begin{bmatrix} 9 & 6 & 2 \\ -7 & -4 & -2 \\ -18 & -12 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -9+6+4 \\ 7-4-4 \\ 18-12-8 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = -1\vec{V}_1 \quad \therefore \boxed{\lambda_1 = -1}$$

$$A\vec{V}_2 = \begin{bmatrix} 9 & 6 & 2 \\ -7 & -4 & -2 \\ -18 & -12 & -4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -18+6+8 \\ 14-4-8 \\ 36-12-16 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 8 \end{bmatrix} = 2\vec{V}_2 \quad \therefore \boxed{\lambda_2 = 2}$$

$$A\vec{V}_3 = \begin{bmatrix} 9 & 6 & 2 \\ -7 & -4 & -2 \\ -18 & -12 & -4 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -18-12+6 \\ 14-8-6 \\ 36+24-12 \end{bmatrix} =$$

$$A\vec{V}_3 = \begin{bmatrix} 9 & 6 & 2 \\ -7 & -4 & -2 \\ -18 & -12 & -4 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -18+12+6 \\ 14-8-6 \\ 36-24-12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\vec{V}_3 \quad \therefore \boxed{\lambda_3 = 0}$$