MAT 239 (Differential Equations), Prof. Swift
Worksheet 26.5, Power Series Solutions

1. (Like WeBWorK problem 4.) The Hermite Equation , $y'' - 2xy' + \lambda y = 0$ is important in physics. Find the *exact* solution of this special case (with $\lambda = 4$) using power series techniques

$$y'' - 2xy' + 4y = 0, \quad y(0) = 1, \; y'(0) = 0$$

$C_0 = 1 \quad C_1 = 0$

2. (Like problem 6.) Find the constants in the series solution $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ of

$$y' + 2xy = 3 + 4x^2, \quad y(0) = 1.$$

$c_0 =$ ______ $\qquad c_1 =$ ______ $\qquad c_2 =$ ______ $\qquad c_3 =$ ______

1. Wait a minute! How do we get an <u>exact</u> solution?
We'll see.

$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

use this $\rightarrow \; = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

$y'' \quad -2x y' \quad +4y = 0$ becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n \underbrace{- 2x \sum_{n=1}^{\infty} n c_n x^{n-1}}_{-2 \sum_{n=1}^{\infty} n c_n x^n} + 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$n=0$ gives same sum

$$\sum_{n=0}^{\infty} \Big[\underbrace{(n+2)(n+1) C_{n+2} - 2n C_n + 4 C_n}_{} \Big] x^n = 0$$

this [ ] = 0 for $n = 0, 1, 2, \ldots$ solve for C_{n+2}

$$\boxed{C_{n+2} = \frac{(2n-4) C_n}{(n+2)(n+1)} \text{ for } n = 0, 1, 2, \ldots}$$ the recurrence relation

Find the solution with $C_0 = 1, C_1 = 0$

$n=0: \; C_2 = \frac{-4}{2 \cdot 1} C_0 = -2 \cancel{C_0}^{1} = -2 \qquad n=1: \; C_3 = \frac{-2}{3 \cdot 2} \cancel{C_1}^{0} = 0$

$n=2: \; C_4 = \frac{0}{4 \cdot 3} C_2 = 0 \qquad n=3: \; C_5 = \frac{2}{5 \cdot 4} \cancel{C_3}^{0} = 0$

$C_6 = C_8 = \ldots = 0 \qquad C_7 = C_9 = \ldots = 0$

only nonzero coefficients are $C_0 = 1$ and $C_2 = -2$. So $\boxed{y = 1 - 2x^2 \text{ is the exact solution}}$

use this!

3. Find the constants in the series solution $\boxed{y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots}$ of

$$y' + 2xy = 3 + 4x^2, \quad y(0) = 1 \quad \therefore C_0 = 1.$$

$c_0 =$ 1 $\qquad c_1 =$ 3 $\qquad c_2 =$ -1 $\qquad c_3 =$ -2/3

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots$$

$$y' = C_1 + 2C_2x + 3C_3x^2 + \cdots$$

$$2x\,y = 2C_0x + 2C_1x^2 + \cdots$$

add:

$$y' + 2xy = C_1 + (2C_2 + 2C_0)x + (3C_3 + 2C_1)x^2 + \cdots$$

Set this equal to the right hand side

$$y' + 2xy = 3 + 0 \cdot x + 4x^2$$

equate coefficients:

$$\boxed{C_1 = 3}, \quad \boxed{2C_2 + 2C_0 = 0}, \quad \boxed{3C_3 + 2C_1 = 4}$$

$C_1 = 3$ ✓ $\qquad 2C_2 + 2(1) = 0 \qquad 3C_3 + 2(3) = 4$

$2C_2 = -2 \qquad 3C_3 = 4 - 6 = -2$

$\boxed{C_2 = -1} \qquad \boxed{C_3 = -\frac{2}{3}}$

See Set 15, Prob 6.
And the awesome videos I made, with links on our web site.