Practice for Bonus Problem

Use Taylor's Theorem with $n=1$ to get an inequality $L < \sqrt{104} < U$, without a calculator.

Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$. We want $f(104)$.
The "nice" input near 104 is 100. $f(100) = \sqrt{100} = 10$.

So, let $x_0 = 100$.

$f(x) = x^{\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$f(100) = (100)^{\frac{1}{2}} = 10$ $\quad f'(100) = \frac{1}{2}(10^2)^{-\frac{1}{2}} = \frac{1}{2}\cdot 10^{-1} = \frac{1}{20}$

Taylor's theorem with $x_0 = 100$, $x = 104$, $n=1$ is

$$f(104) = f(100) + f'(100)(104-100) + \frac{f''(\xi)}{2}(104-100)^2.$$

or

$$\sqrt{104} = 10 + \frac{1}{20}\cdot 4 + \frac{-\frac{1}{4}\xi^{-\frac{3}{2}}}{2}\cdot 4^2 = 10 + \frac{1}{5} - 2\xi^{-\frac{3}{2}}$$

for some ξ between 100 and 104.
$\xi^{-3/2}$ is decreasing with ξ, so $-2\xi^{-\frac{3}{2}}$ is increasing, and

$$-2\cdot(100)^{-\frac{3}{2}} < -2\xi^{-\frac{3}{2}} < -2\cdot(104)^{-3/2} \text{ for } 100<\xi<104.$$

$$-2\cdot 10^{-3} < -2\xi^{-\frac{3}{2}} < \frac{-2}{\sqrt{104}\cdot 104}$$

Now, we need to estimate $\frac{-2}{\sqrt{104}\cdot 104}$ without a calculator. No fair evaluating $\sqrt{104}$ on the calculator!

There are at least 2 ways to proceed without a calculator.

Method 1. (Punt)

Don't try to be too fancy. Note that $\frac{-2}{\sqrt{104}\cdot 104}$ is negative.

So $\frac{-2}{1000} < -2\xi^{-3/2} < 0$, and it follows that

$10 + \frac{1}{5} - \frac{2}{1000} < \sqrt{104} < 10 + \frac{1}{5}$, or $\boxed{10.198 < \sqrt{104} < 10.2}$

Method 2. (use a $\sqrt{\ }$ you know)

If we replace $\sqrt{104}$ with $\sqrt{121} = 11$, we get a larger denominator, and a fraction closer to 0.

So $\frac{-2}{1000} < -2\xi^{-3/2} < \frac{-2}{\sqrt{104}\cdot 104} < \frac{-2}{\sqrt{121}\cdot 104} = \frac{-2}{11\cdot 104}$

or $\frac{-2}{1000} < -2\xi^{-3/2} < \frac{-1}{11\cdot 52} = \frac{-1}{572}$

So $10 + \frac{1}{5} - \frac{2}{1000} < \sqrt{104} < 10 + \frac{1}{5} - \frac{1}{572}$

or $\boxed{10.198 < \sqrt{104} < 10.2 - \frac{1}{572}} < \underbrace{10.1983}_{\text{rounded up, using calculator.}}$

Note that $\sqrt{104} = 10.19839\ldots$

$10.19825 < 10.2 - \frac{1}{572} < 10.1983$, using calculator.

So, the inequality is true!