MAT 461, Prof. Swift
The Fourier Cosine Series of $\sin(\pi x/L)$

Problem: Find the Fourier Cosine Series (FCS) of $\sin(\pi x/L)$ on the interval $0 \leq x \leq L$, using equation (3.4.13) instead of integration.

Solution: First write the FCS with unknown coefficients.

$$\sin(\pi x/L) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L) \tag{1}$$

Since $\sin(\pi x/L)$ is a continuous function and its derivative is piecewise smooth, we can differentiate term by term to get

$$\frac{\pi}{L}\cos(\pi x/L) \sim 0 + \sum_{n=1}^{\infty} A_n \frac{-n\pi}{L}\sin(n\pi x/L)$$

We can solve this for the FSS of $\cos(\pi x/L)$, in terms of the still unknown A_n.

$$\cos(\pi x/L) \sim \sum_{n=1}^{\infty}(-nA_n)\sin(n\pi x/L) \tag{2}$$

Now, define $f(x) = \cos(\pi x/L)$. Since f' is piecewise smooth, and f is continuous, we can apply equation (3.4.13) from the book,

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}B_n + \frac{2}{L}((-1)^n f(L) - f(0))\right]\cos(n\pi x/L). \tag{3}$$

In our application, $f(x) = \cos(\pi x/L)$, so $f'(x) = -\pi/L\sin(\pi x/L)$, $f(0) = 1$, $f(L) = -1$, and $B_n = -nA_n$. Equation (3) becomes

$$\frac{-\pi}{L}\sin(\pi x/L) \sim \frac{1}{L}[-1-1] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}(-nA_n) + \frac{2}{L}((-1)^n(-1) - 1)\right]\cos(n\pi x/L)$$

Solving this previous equation for $\sin(\pi x/L)$ gives

$$\sin(\pi x/L) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[n^2 A_n + \frac{2}{\pi}((-1)^n + 1)\right]\cos(n\pi x/L)$$

Comparing this most recent equation with equation (1) gives

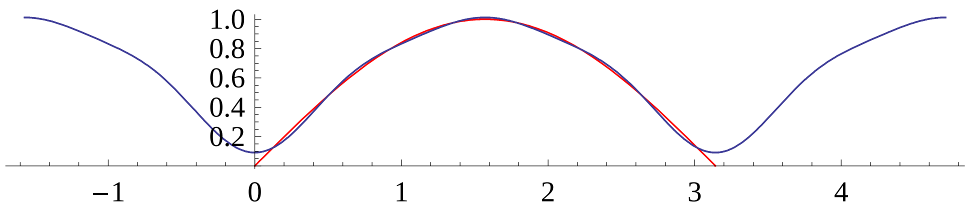
$$A_0 = \frac{2}{\pi}, \quad A_n = n^2 A_n + \frac{2}{\pi}((-1)^n + 1) \text{ for } n \geq 1.$$

This can be solved for $A_n = \frac{2((-1)^n+1)}{\pi(1-n^2)}$ when $n \geq 2$. (This method does not prove that $A_1 = 0$, since the equation if $A_1 = A_1 + 0$, but we know that $A_1 = 0$ since $\sin(\pi x/L)$ is even about $x = L/2$.) Therefore we get the FCS of $\sin(\pi x/L)$, and simplify it using the fact that $(-1)^n + 1 = 2$ if n is even (*i.e.* $n = 2k$), and $(-1)^n + 1 = 0$ if n is odd.

$$\sin(\pi x/L) \sim \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n + 1}{1 - n^2}\cos(n\pi x/L) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{1 - (2k)^2}\cos(2k\pi x/L).$$

The original problem is solved, but now let's look at some figures and explore the consequences of the previous calculation.

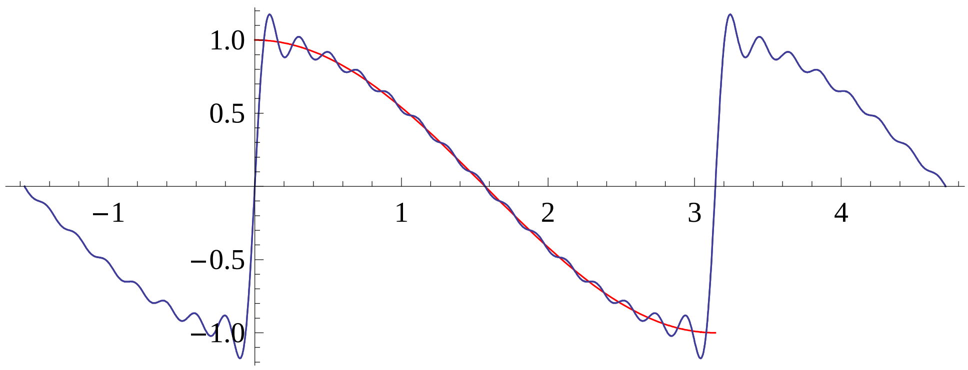
The next figure shows a partial sum of the FCS of $\sin(x)$ on the interval $0 \leq x \leq \pi$. (That is, $L = \pi$.) The partial sum plotted has only four nonzero terms. Note that the FCS is continuous.



As a bonus, we can also get the FSS of $\cos(\pi x/L)$, using equation (2).

$$\cos(\pi x/L) \sim \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{-n((-1)^n+1)}{1-n^2}\sin(n\pi x/L) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{k}{(2k)^2-1}\sin(2k\pi x/L).$$

The next figure shows a partial sum of the FSS of $\cos(x)$ on the interval $0 \leq x \leq \pi$. The partial sum has 15 nonzero terms. Since the FSS is not continuous, the convergence is not so good and you see the Gibbs phenomenon.



These figures were produced by the *Mathematica* notebook `FCSofSin.nb`, available on our class web site.