MAT 461. Proof of the orthogonality relation, equation (2.3.32).

$\boxed{\begin{array}{l}\underline{\text{Fact}}.\ \text{If } n \text{ and } m \text{ are positive integers, then} \\ I_{nm} := \displaystyle\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \end{cases}\end{array}}$

<u>Proof</u>: Let $u = \frac{\pi}{L}x$. Then $du = \frac{\pi}{L}dx$ or $dx = \frac{L\,du}{\pi}$, and the limits are: $\begin{array}{l} x=L \Rightarrow u = \frac{\pi}{L}\cdot L = \pi \\ x=0 \Rightarrow u=0 \end{array}$. Thus, the integral is

$$I_{nm} = \frac{L}{\pi}\int_0^\pi \sin(nu)\sin(mu)\,du.$$

use the trig identity $\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$I_{nm} = \frac{L}{2\pi}\int_0^\pi \left[\cos((n-m)u) - \cos((n+m)u)\right]du$$

Case I: $n=m$.

$$I_{nn} = \frac{L}{2\pi}\int_0^\pi \left[1 - \cos(2nu)\right]du = \frac{L}{2\pi}\left[u - \frac{\sin(2nu)}{2n}\right]\Bigg|_0^\pi$$

$$= \frac{L}{2\pi}\left[\pi - \frac{\overbrace{\sin(2n\pi)}^{0}}{2n} - \left(0 - \frac{\sin 0}{2n}\right)\right] = \frac{L}{2\pi}\cdot\pi = \frac{L}{2}.$$

Case II: $n \neq m$

$$I_{nm} = \frac{L}{2\pi}\left[\frac{\sin((n-m)u)}{n-m} - \frac{\sin((n+m)u)}{m+n}\right]_0^\pi = 0$$

The evaluations $\sin((n-m)\pi)$, etc. are all $0$, since sine of an integer multiple of $\pi$ is $0$.