MAT 661 (Applied Mathematics)
Homework # 1 due before class Monday, August 24, 2020
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1. Maxwell's equations in free space, using Gaussian Units, are

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field.

(a) Show that the electric field satisfies the wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}, \tag{1}$$

where the Laplacian of a vector field means to take the Laplacian of each component. That is, the vector Laplacian of $\mathbf{E} = (E_1, E_2, E_3)$ is $\nabla^2 \mathbf{E} = (\nabla^2 E_1, \nabla^2 E_2, \nabla^2 E_3)$. You may assume that all functions are diffentiable as often as needed (they are C^∞). You may also use, without proof, a vector identity for $\nabla \times (\nabla \times \mathbf{A})$ that you can find on-line or elsewhere. (I searched for "vector calculus identities").

(b) Verify that $\mathbf{E}(x, y, z, t) = E \sin(kz - ckt)\hat{\mathbf{i}}$ solves the wave equation (1) for any scalar constants E and k. Here $\hat{\mathbf{i}} = (1, 0, 0)$ is the unit vector in the x-direction. Find the magnetic field for this light wave traveling in the z-direction. Try to sketch the electric and magnetic fields.

(c) Light is a *transverse wave*. This means that the direction of the electric and magenetic fields are perpendicular to the direction of propogation. If the z is replaced by x in the solution in part (a), we get $\mathbf{E}(x, y, z, t) = E \sin(kx - ckt)\hat{\mathbf{i}}$, which is not a transverse wave. While this new function satisfies the wave equation (1) it is not a solution to Maxwell's equations. Why?

2. A useful tool in studying ODEs and PDEs that come from physics is non-dimensionalization. The driven, damped pendulum has this equation of motion $F = ma$, actually $ma = F$, for the tangential acceleration:

$$m\ell \frac{d^2\theta}{dt^2} = -\gamma \frac{d\theta}{dt} - mg\sin(\theta) + F\cos(\omega_d t).$$

The unit of mass m is kilograms. We write this statement as $[m] = \text{kg}$. Similarly, $[\ell] = \text{m}$ (meters), $[t] = \text{s}$ (seconds). This is the MKS system: (meters, kilograms, seconds). Angles like θ, measured in radians, are unitless, so $[\theta] = 1$. The acceleration of gravity g has units $[g] = \text{m/s}^2$. Define $\omega_0 = \sqrt{g/\ell}$. Show that you can write the equations in terms of the dimensionless time variable $\omega_0 t = \bar{t}$ as

$$\frac{d^2\theta}{d\bar{t}^2} = -c\frac{d\theta}{d\bar{t}} - \sin(\theta) + \rho\cos(\bar{\omega}\bar{t}).$$

Find the constants dimensionless constants c and ρ and $\bar{\omega}$ in terms of the original constants m, ℓ, γ, g, and F.

Read Chapter 1, sections 1 and 2, and do problems 1.1, 1.6, 2.1, 2.2, 2.5