MAT 661 (Applied Mathematics), Prof. Swift
Homework # 2

When studying PDEs, it is important to know the solutions to some ODEs "by inspection". It is not so much that you memorize these solutions by rote, but that you internalize the solution and the verification "flashes before your eyes" as you write down the solution. Here they are:
Assume k, A, y_0 and v_0 are constants. Recall that an Initial Value Problem (IVP) is an ODE with initial conditions.

The IVP $\dfrac{dy}{dt} = ky,\ y(0) = y_0$ has the solution $y = y_0 e^{kt}$

The IVP $\dfrac{dy}{dt} = k(y - A),\ y(0) = y_0$ has the solution $y = A + (y_0 - A)e^{kt}$

The IVP $\dfrac{d^2y}{dt^2} = -k^2 y,\ y(0) = y_0,\ y'(0) = v_0$ has the solution $y = y_0 \cos(kt) + v_0/k \sin(kt)$

The IVP $\dfrac{d^2y}{dt^2} = k^2 y,\ y(0) = y_0,\ y'(0) = v_0$ has the solution $y = y_0 \cosh(kt) + v_0/k \sinh(kt)$. Recall that $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh(x) = (e^x - e^{-x})/2$.

Use these solutions to solve these applied problems.

1. The population of pigs on an island grows exponentially. It the population starts at 500 and is initially growing at a rate of 100 per year, what is the population $P(t)$ after t years?

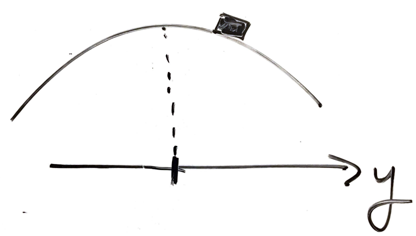
2. You take out a loan for $\$20,000$ to buy a car, at 4.5% per year interest rate. What is your monthly payment P so that you pay off the loan in 5 years?
Hint: If $B(t)$ is the ballance you owe the bank after t nears, and the payment is assumed to be continuous in time, the balance satisfies

$$\frac{dB}{dt} = 0.045B - 12P$$

3. The position of a mass on a spring, $y(t)$, satisfies the ODE

$$m\frac{d^2y}{dt^2} = -ky$$

where m is the mass and k is the spring constant. (We are neglecting friction.) Define $\omega_0 = \sqrt{k/m}$, and give all the following answers in terms of ω_0, not in terms of m or k separately. What are the units of ω_0? Find the solution for $y(t)$ if the initial position is 0 and the initial velocity is v_0. What is the period of the oscillation? What is the amplitude of the oscillation?



4. A frictionless puck near the top of rounded ice mountain satisfies the equation of motion

$$\frac{d^2y}{dt^2} = \alpha y$$

where α is a positive constant. If the initial position is y_0, what should the initial velocity v_0 be so that the puck approaches the top of the peak ($y = 0$) as t approaches infinity?

5. (a) Solve the IVP $y'' + 16y = 0,\ y(1) = 3,\ y'(1) = 2$ by inspection.

(b) For what values of λ does the Boundary Value Problem (BVP) $y'' = \lambda y,\ y(0) = y(1) = 0$ have a nontrivial solution? (Show that there are no nontrivial solutions for $\lambda > 0$.)

6. Find the transcendental equation for λ such that the BVP $y'' = \lambda y,\ y(0) = 0,\ y(1) - y'(1) = 0$ has a nontrivial solution. This is the analog of the characteristic polynomial of a matrix. Use Mathematica or MATLAB to approximate the three largest values of λ and graph a corresponding "eigenfunction" y for each of these "eigenvalues".

7. Same as problem 6, except change the boundary condition to
$y(0) = 0,\ 2y(1) - y'(1) = 0$.

Read Chapter 1, sections 3 and 4, Chapter 2, section 1.

Problems 3.2, 3.4, 3.6 (Make a computer plot of at least one of the problems in this set.)

Problems 4.2, 4.3, 4.4, 4.5, 4.7, 4.10