Problem II 4.2-type problem.
Verify that the curve C is an integral curve of V, and derive a formula for infinitely many integral surfaces of V containing C.

V = (x, -y, 0), C: x = t, y = -1/t, z = 0 for all t > 0.

Answer: The surface with equation
F(xy, z) = 0
is an integral surface for any smooth function F:R^2 -> R that satisfies F(-1,0) = 0.

I will show a “model solution” in this notebook. I will also illustrate this result, showing the surfaces obtained by a “standard” choice of F.

Solution:
Two Functionally Independent First Integrals (FIFIs) for V are

```
In[ ]:= u1[x_, y_, z_] := x y
u2[x_, y_, z_] := z
```

The hardest part of the problem is determining these 2 FIFIs and verifying that they are indeed functionally independent by computing the cross product of the gradients.

To show that C is an integral curve, compute
U1(t) = u1(t, -1/t, 0) = (t)(-1/t) = -1 (for all t > 0) and
U2(t) = u2(t, -1/t, 0) = 0 (for all t > 0)

Since U1(t) and U2(t) are constant functions, the curve C is an integral curve of V.

Any smooth function F:R^2 → R will give a first integral F(xy, z) of V by Theorem 3.1. This integral surface will contain C if and only if F(-1, 0) = 0, as shown in the proof of Theorem 4.3.
(End of the “model solution”.)

Now lets illustrate this result with a one-parameter family of functions F that satisfy F(-1, 0) = 0, out of the unfathomably largely infinitely many such functions.

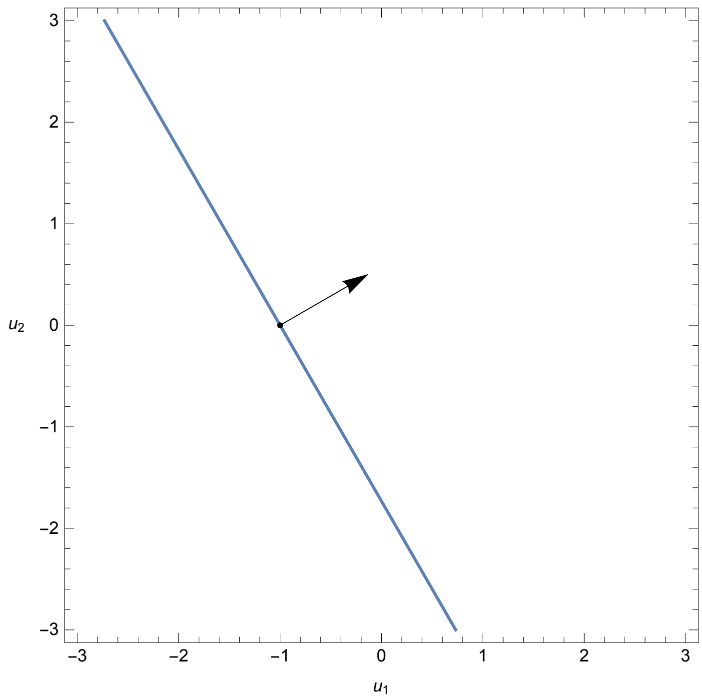
Here’s a tangent for you: If we assume the continuum hypothesis, the cardinality of the set of functions F defined below is Aleph-1.   I don’t know set theory very well, but I think the cardinality of all functions F:R^2 → R that satisfy F(-1,0) = 0 is Aleph-2. (That’s what I mean by unfathomably largely infinitely many such functions.)

My choice for a function F:R^2 → R that satisfies F(c1, c2) = 0 has a parameter θ. So I actually define a function F:R^5 -> R, with c1, c2, and θ as the last three arguments of the function. The set F(u1, u2, c1, c2, θ) = 0 is a line in through the point (c1, c2) that is perpendicular to the vector (cos(θ), sin(θ)).

```
In[•]:= F[u1_, u2_, c1_, c2_, θ_] := (u1 - c1) Cos[θ] + (u2 - c2) Sin[θ]

In[•]:= θ = π / 6;
        {c1, c2} = {-1, 0};
        ContourPlot[F[u1, u2, c1, c2, θ] == 0, {u1, -3, 3}, {u2, -3, 3},
         Epilog → {Arrow[{{c1, c2}, {c1, c2} + {Cos[θ], Sin[θ]}}], Point[{c1, c2}]},
         FrameLabel → {"u₁", "u₂"}, RotateLabel → False]
```

Out[•]=



Now we plot the family of integral surfaces that go through C: x = t, y = -1/t, z = 0,
depending on the parameter θ. We have already determined that c1 = -1 and c2 = 0.
You must first run the cells where u1, u2, and F are defined, before running this cell. I like to select all (Ctrl A) and run (Shift-ENTER, or bottom-right ENTER on some keyboards).

You can use the slider to change θ.
Clicking the + to the right of the slider allows you to see the value of θ and also animate the figure!

```
In[•]:= Manipulate[
         L = 3;  (* -L ≤ x,y ≤ L *)
         H  = 2; (* -H ≤ z ≤ H *)
         c1 = -1;
         c2 = 0;
         integralCurve = Graphics3D[Tube[Table[{t, -1/t, 0}, {t, 1/L, L, 1/(5 L)}], .07]];
         integralSurface = ContourPlot3D[F[u1[x, y, z], u2[x, y, z], c1, c2, θ] == 0,
           {x, -L, L}, {y, -L, L}, {z, -H, H}, Mesh → False];
         Show[integralSurface, integralCurve, AxesLabel → {"x", "y", "z"},
          BoxRatios → Automatic]
         , {{θ, 2.2}, 0, π}]
```

Now do the same thing for the integral curve C: x = 0, y = -t, z = 0, t > 0.

In this case, c1 = u1(0,-t, 0) = 0, and c2 = u2(0, -t, 0) = 0.

```
In[ ]:= Manipulate[
 L = 3;  (* -L ≤ x,y ≤ L *)
 H  = 2; (* -H ≤ z ≤ H *)
 c1 = 0;
 c2 = 0;
 integralCurve = Graphics3D[Tube[Table[{0, -t, 0}, {t, 0, L, L}], .07] ];
 integralSurface = ContourPlot3D[F[u1[x, y, z], u2[x, y, z], c1, c2, θ] == 0,
   {x, -L, L}, {y, -L, L}, {z, -H, H}, Mesh → False];
 Show[integralSurface, integralCurve, AxesLabel → {"x", "y", "z"},
  BoxRatios → Automatic]
 , {{θ, 2.2}, 0, π}]
```