§1.3 Corollary 4.

If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then $e^A = e^a \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$

Proof (Different from book)

Diagonalize A over $\mathbb{C}$!

$\det(A-\lambda I) = (a-\lambda)^2 + b^2 = 0$, so $\lambda_\pm = a \pm ib$

Let $\lambda_+ = a+ib$. $A - \lambda_+ I = \begin{bmatrix} -ib & -b \\ b & -ib \end{bmatrix}$

One choice of eigenvector is $\vec{V}_+ = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Since A is real, it follows that $\vec{V}_- = \begin{bmatrix} 1 \\ i \end{bmatrix}$, the complex conjugate of $\vec{V}_+$.

Let $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Then $P^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$

Then $B := \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix} = P^{-1}AP$ and $A = PBP^{-1}$.

By §1.3 Corollary 1, $e^A = P \begin{bmatrix} e^{a+ib} & 0 \\ 0 & e^{a-ib} \end{bmatrix} P^{-1}$

$$\text{So } e^A = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} e^a \begin{bmatrix} e^{ib} & 0 \\ 0 & e^{-ib} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{e^a}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{ib} & 0 \\ 0 & e^{-ib} \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$= \frac{e^a}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{ib} & ie^{ib} \\ e^{-ib} & -ie^{-ib} \end{bmatrix}$$

$$= \frac{e^a}{2} \begin{bmatrix} e^{ib} + e^{-ib} & ie^{ib} - ie^{-ib} \\ -ie^{ib} + ie^{-ib} & e^{ib} + e^{-ib} \end{bmatrix}$$

$$= e^a \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$$

Note: $e^{ib} = \cos(b) + i\sin(b)$

so

$\frac{e^{ib} + e^{-ib}}{2} = \cos(b)$

$\frac{e^{ib} - e^{-ib}}{2i} = \sin(b)$

or $\frac{-ie^{ib} + ie^{-ib}}{2} = \sin(b)$