COMPLEX EIGENVALUES

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This handout concerns the solution to $\dot{\mathbf{x}} = A\mathbf{x}$, where A is a square matrix with real entries and simple complex (non-real) eigenvalues.

For simplicity, assume that A is 2×2. We will do other cases by example. Following Perko's book, assume that

$$A\mathbf{w} = (a + ib)\mathbf{w},$$

with $a, b \in \mathbb{R}$ and $b > 0$. Since A has real entries, $A\overline{\mathbf{w}} = (a - ib)\overline{\mathbf{w}}$. We can write the (complex) eigenvector $\mathbf{w}$ in terms of its real and imaginary parts as

$$\mathbf{w} = \mathbf{u} + i\mathbf{v}$$

where $\mathbf{u}$ and $\mathbf{v}$ both have real entries. There is quite a bit of freedom in choosing $\mathbf{u}$ and $\mathbf{v}$, since $\alpha\mathbf{w}$ is an eigenvector of A for any $\alpha \in \mathbb{C}$.

We can find the general (real-valued) solution to $\dot{\mathbf{x}} = A\mathbf{x}$ by following the usual procedure for distinct eigenvalues:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{w} & \overline{\mathbf{w}} \end{bmatrix} \begin{bmatrix} e^{(a+ib)t} & 0 \\ 0 & e^{(a-ib)t} \end{bmatrix} \begin{bmatrix} \gamma \\ \overline{\gamma} \end{bmatrix} = \mathbf{w}e^{(a+ib)t}\gamma + \text{c.c.}$$

where the arbitrary constant is $\gamma \in \mathbb{C}$, and the "+ c.c." means to add the complex conjugate of the preceding expression. The second constant is $\overline{\gamma}$ to ensure that $\mathbf{x}(t)$ is real-valued.

To obtain a nice expression for the general solution, let $\gamma = -ir_{\text{o}}e^{i\theta_{\text{o}}}/2$, where $r_{\text{o}}, \theta_{\text{o}} \in \mathbb{R}$. The general solution is

$$\mathbf{x}(t) = r_{\text{o}}e^{at}\left(\mathbf{v}\cos(bt + \theta_{\text{o}}) + \mathbf{u}\sin(bt + \theta_{\text{o}})\right). \tag{1}$$

with the two arbitrary constants r_{o} and θ_{o}.

While the general solution in equation (1) is good for understanding the trajectories, it is not the best form for getting a solution to the IVP $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_{\text{o}}$. To solve the IVP, first let $\gamma = -i(c_1 + ic_2)/2$, with $c_1, c_2 \in \mathbb{R}$. When the dust settles, the general solution can be written as

$$\mathbf{x}(t) = e^{at}\begin{bmatrix} \mathbf{v} & \mathbf{u} \end{bmatrix} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where the two arbitrary constants are c_1 and c_2. Following Perko, define $P = [\mathbf{v}\ \mathbf{u}]$. Since $\mathbf{x}(0) = \mathbf{x}_{\text{o}}$, the constants satisfy $P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_{\text{o}}$, which can be solved to give $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}_{\text{o}}$ so the solution to the IVP is

$$\mathbf{x}(t) = e^{at}P\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} P^{-1}\mathbf{x}_{\text{o}}, \tag{2}$$

and we have finally reproduced Perko's solution.

Now, consider the ODE in the $\mathbf{y}$ coordinates defined by $\mathbf{x} = P\mathbf{y}$. The ODE is

$$\dot{\mathbf{y}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{y}$$

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with initial conditions $\mathbf{y}_o = P^{-1}\mathbf{x}_o = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. The constants r_o and θ_o in the general solution (1) are related to c_1 and c_2 by $c_1 + ic_2 = r_o e^{i\theta_o}$, or

$$\begin{aligned} c_1 &= r_o \cos(\theta_o) \\ c_2 &= r_o \sin(\theta_o). \end{aligned}$$

Thus, r_o and θ_o are the polar coordinates of the initial point $\mathbf{y}_o = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in the new coordinates. The solution in the $\mathbf{y}$ system, in polar coordinates, is $r(t) = r_o e^{at}$ and $\theta(t) = bt + \theta_o$. The trajectories lie on logarithmic spirals $r = r_o e^{a(\theta-\theta_o)/b}$.

Example: Suppose that A is a real 2×2 matrix, and $A\mathbf{w} = 2i\mathbf{w}$, where $\mathbf{w} = \begin{bmatrix} 1+4i \\ 3 \end{bmatrix}$. We can write down the general solution by inspection, using equation (1):

$$\mathbf{x}(t) = r_o \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} \cos(2t + \theta_o) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \sin(2t + \theta_o) \right).$$

Note that the P matrix does not need to be inverted, and no matrix multiplication is needed. If you want to solve the initial value problem, equation (2) is better, but the general solution based on equation (1) is easier to understand geometrically.

Note that we can figure out A in this case by using $P = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Then some matrix algebra shows that

$$A = PBP^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{17}{6} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Do not be alarmed during an exam if I present the information about a matrix in a form similar to the first sentence of this example. I am trying to be nice to you by giving you the eigenvalues and eigenvectors on platter.

Example: For the example on pages 29 and 30 of Perko's book, the general solution can again be obtained by inspection. Here it is natural to use the constants r_1 and θ_1 for the part of the solution associated with $\lambda_1 = 1 + i$, and r_2 and θ_2 for the part coming from $\lambda_2 = 2 + i$.

$$\mathbf{x}(t) = r_1 e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cos(t + \theta_1) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \sin(t + \theta_1) \right) + r_2 e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cos(t + \theta_2) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \sin(t + \theta_2) \right).$$

Example: In the example on pages 30 and 31, the real matrix A satisfies $A\mathbf{v}^{(1)} = -3\mathbf{v}^{(1)}$ and $A\mathbf{w}^{(2)} = (2+i)\mathbf{w}^{(2)}$, where

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w}^{(2)} = \mathbf{u}^{(2)} + i\mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}.$$

The general solution in this case is

$$\mathbf{x}(t) = c_1 e^{-3t}\mathbf{v}^{(1)} + r_o e^{2t} \left(\mathbf{v}^{(2)} \cos(t + \theta_o) + \mathbf{u}^{(2)} \sin(t + \theta_o) \right),$$

where I have used the constants r_o and θ_o for the complex eigenvalue. It is clear from this vector form of the general solution that the stable subspace $E^s = \text{span}\{\mathbf{v}^{(1)}\}$ and the unstable subspace $E^u = \text{span}\{\mathbf{u}^{(2)}, \mathbf{v}^{(2)}\}$ are invariant. (That is, an initial condition in one of these invariant subspaces leads to a solution that stays in the invariant subspace for all time.)