MAT 665 Example - Complex Eigenvalues

-Jim Swift

- Find the general solution to $\dot{\vec{x}} = A\vec{x}$, with

$$A = \begin{bmatrix} 1 & -4 \\ 2 & -3 \end{bmatrix}$$

- Sketch the phase portrait in the $\vec{y}$ and $\vec{x}$ planes.

Step 1: compute eigenvalues:

$$\tau = -2 \quad \left(\tfrac{\tau}{2} = -1\right), \quad \delta = -3 + 8 = 5$$

$$\lambda = -1 \pm \sqrt{(-1)^2 - 5} = -1 \pm \sqrt{-4}, \quad \boxed{\lambda = -1 \pm 2i}$$

Step 2: compute one eigenvector

evec. of $\lambda_+ = -1 + 2i$

$$A - \lambda_+ I = \begin{bmatrix} 2-2i & -4 \\ 2 & -2-2i \end{bmatrix} \quad \text{so} \quad \vec{w} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{u}} + i \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}}$$

Using formula, the general solution is

$$\boxed{\vec{x}(t) = r_0 e^{-t}\left(\cos(2t+\theta_0)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(2t+\theta_0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}$$

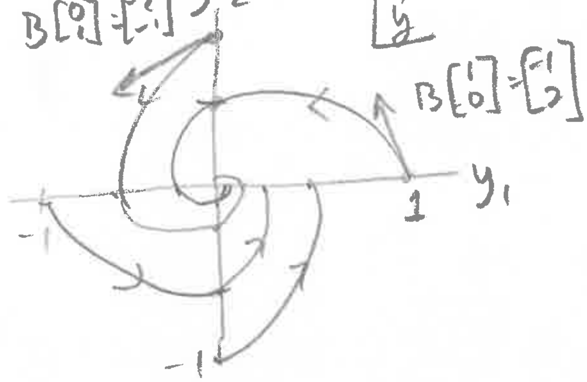
In the $\vec{y}$-coordinates, $\dot{\vec{y}} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}\vec{y}$, and

$$\left.\begin{aligned} r(t) &= r_0 e^{-t} \\ \theta(t) &= \theta_0 + 2t \end{aligned}\right\} \text{ eliminate } t: \quad r(\theta) = r_0 e^{-\left(\frac{\theta - \theta_0}{2}\right)}$$

make a table

$\theta - \theta_0$	r/r_0
0	1
$\frac{\pi}{2}$	$e^{-\frac{\pi}{4}} = 0.46\ldots$ $(= R)$
π	$e^{-\frac{\pi}{2}} = 0.21\ldots = R^2$
$\frac{3\pi}{2}$	0.09 $(= R^3)$
2π	0.04 $(= R^4)$

I define "ratio" R



Phase portrait for $A = \begin{bmatrix} 1 & -4 \\ 2 & -3 \end{bmatrix}$, continued.

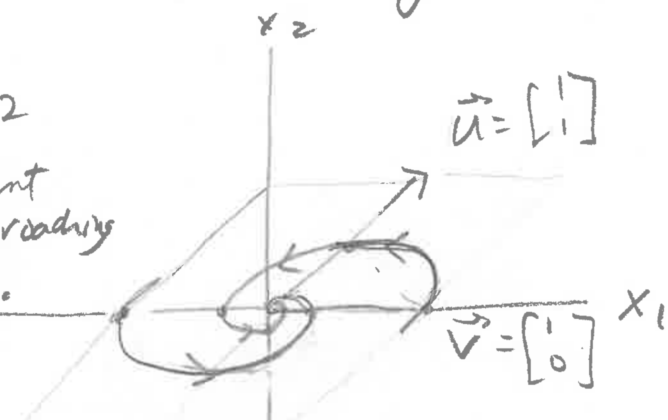
Sketch $\vec{u}$ and $\vec{v}$.

Flow goes from $\vec{v}$ to $\vec{u}$, and for every quarter turn the radius factor e^{at} is $R = 0.46\ldots$ $\qquad \mapsto$ is advance by $\Delta t = \frac{\pi}{4}$

That is, $\vec{x} = \vec{v} \mapsto \vec{x} = 0.46\,\vec{u} \mapsto -0.21\,\vec{v} \mapsto -0.09\,\vec{u}$

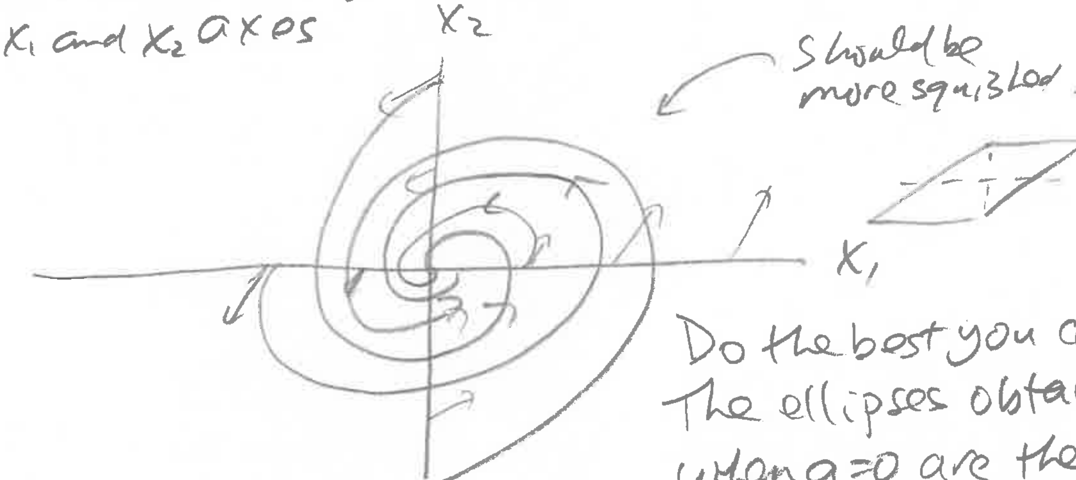
We can also sketch the parallelogram with vertices $\pm\vec{u} \pm \vec{v}$, which is the image of the unit square in $\vec{y}$ under $\vec{x} = P\vec{y} = [\vec{v}\ \vec{u}]\,\vec{y}$.

Note: when $\lambda = a \pm ib$ with $a > 0$ (like the 2 phase portraits you drew), you might want to sketch solution approaching $\vec{0}$ as t decreases.



Note: $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Hint: After you did this, a good phase portrait can be obtained by sketching directions of $A\vec{x}$ along the x_1 and x_2 axes



Should be more squished!

Do the best you can! The ellipses obtained when $a = 0$ are the easiest to sketch.