MAT 665, Differential Equations, Prof. Swift
Formula sheet from MAT 239. Notes with • for MAT 665

First order ODEs: • $\frac{dy}{dx} = f(x, y)$ (which is used here) or $\frac{dy}{dt} = f(t, y)$ or $\frac{dx}{dt} = f(t, x)$.

Separation of variables is the simplest method.

The standard form for a first order linear ODE is $y' + p(x)y = g(x)$. This can be solved using the integrating factor $\mu(x) = \exp(\int p(x)\, dx)$.

A first order ODE in differential form, $P(x, y)dx + Q(x, y)dy = 0$, is exact if and only if $P_y(x, y) = Q_x(x, y)$. If the ODE is exact, the solutions are on level curves $F(x, y) = C$, with $F_x = P$ and $F_y = Q$.

Linear ODEs of order 2 or higher

A linear homogeneous ODE can be written as $L[y] = 0$, where $L = p_n(t)\dfrac{d^n}{dt^n} + \cdots + p_1(t)\dfrac{d}{dt} + p_0(t)$ is an n order linear operator. The ODE is usually written as $a_n(t)\dfrac{d^n y}{dt^n} + \cdots + a_1(t)\dfrac{dy}{dt} + a_0(t)y = 0$ The general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$, a linear combination of n linearly independent solutions.

A real root r of the characteristic equation corresponds to a solution $y = e^{rt}$ of a linear, homogeneous ODE with constant coefficients (LHODECC).

A complex conjugate pair of roots $r = a \pm ib$ of the characteristic equation corresponds to two solutions $y = e^{at}\cos(bt)$ and $y = e^{at}\sin(bt)$of the LHODECC.

Repeated roots introduce factors of t to get linearly independent solutions to a LHODECC

The general solution to a nonhomogeneous linear ODE $L[y] = g(t)$, for a fixed linear operator L and function g, is $y = y_h + y_p$, where y_h is the general solution to the associated homogeneous ODE, and y_p is one particular solution to the nonhomogeneous ODE.

• By analogy, the general solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ with a fixed $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \in \mathbb{R}^n$, where $\mathbf{x}_h$ is the general solution to the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_p$ is one particular solution to the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$.

Systems of ODEs

These formula concern the solution to $\mathbf{x}' = A\mathbf{x}$, where A is a 2×2 matrix with constant, real entries. The eigenvalues of A satisfy $\det(\mathrm{A} - \lambda \mathrm{I}) = 0$, and the associated eigenvectors satisfy $A\mathbf{v} = \lambda\mathbf{v}$, or $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Case 1. A has real, distinct eigenvalues, $\lambda_1 \neq \lambda_2$.

The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. The constants c_1 and c_2 are determined by the initial condition.

Case 2. A has complex eigenvalues $\lambda = a \pm ib$. To solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, first compute $\mathbf{x}_1 = (A - aI)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{at}\left(\mathbf{x}_0 \cos(bt) + \mathbf{x}_1 \tfrac{1}{b}\sin(bt)\right)$$

• Most texts use complex eigenvectors. Let $\lambda_1 = a + ib$, and $\mathbf{v}_1 = Re(\mathbf{v}_1) + i\,Im(\mathbf{v}_1)$. Two linearly independent solutions are $Re\big(\exp(\lambda_1 t)\mathbf{v}_1\big)$ and $Im\big(\exp(\lambda_1 t)\mathbf{v}_1\big)$. A calculation shows that the general solution can be written as

$$\mathbf{x}(t) = c_1 e^{at}\big(Re(\mathbf{v}_1)\cos(bt) - Im(\mathbf{v}_1)\sin(bt)\big) + c_2 e^{at}\big(Im(\mathbf{v}_1)\cos(bt) + Re(\mathbf{v}_1)\sin(bt)\big).$$

Case 3. A has repeated, real eigenvalues, $\lambda_1 = \lambda_2$. To solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, first compute $\mathbf{x}_1 = (A - \lambda_1 I)\mathbf{x}_0$. Then the solution to the IVP is

$$\mathbf{x}(t) = e^{\lambda_1 t}\left(\mathbf{x}_0 + \mathbf{x}_1 t\right)$$

• In MAT 665 we will fully develop the solutions of linear ODEs with repeated roots.

Nonhomogeneous: The general solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ is $\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$, where $\mathbf{x}_h(t)$ is the general solution to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}_p(t)$ is one particular solution to the nonhomogeneous system.