Solutions to Problem Set 1 of MAT 665

MAT 239, Differential Equations, Prof. Swift
Spring 2007 Final Exam: Wednesday, May 9, 7:30 am - 9:30

Name: key

Four pages of notes with writing on both sides are allowed. A graphing calculator is expected. There are no difficult calculations or hard integrals if you use the correct approach. If you have a question, ask me. **Show your work!**

1. Solve the ODE. (You do not need to show your work *on this problem.* Just write down the solution, if you know it.)

(a) $\frac{dy}{dt} = 3y,\ y(0) = 2.$ $y = 2e^{3t}$

(b) $\frac{d^2y}{dt^2} = -9y$ $y = c_1\cos(3t) + c_2\sin(3t)$

(c) $\frac{dy}{dt} = 2(y-1)$ $y = 1 + ce^{2t}$ or $y = 1 + (y_0 - 1)e^{2t}$

(d) $\frac{d^3y}{dt^3} = 0$ $y = c_1 + c_2 t + c_3 t^2$

(e) $\frac{d^2y}{dt^2} + t^2\frac{dy}{dt} + y^2 = 0,\ y(0) = 0,\ y'(0) = 0.$ $y = 0$

5 (2 pts)

2. Find the general solution to $y' + \frac{2}{t}y = t$.

$p(t) = \frac{2}{t}$

$\mu(t) = \exp\int \frac{2}{t}\,dt = \exp(2\ln t)$

$= \exp(\ln t^2) = t^2$

so $t^2y' + 2ty = t^3$

$(t^2y)' = t^3$

$t^2y = \frac{t^4}{4} + C$

$$\boxed{y = \frac{t^2}{4} + \frac{c}{t^2}}$$

3. (a) Find an explicit solution to the IVP $\frac{dy}{dt} = y^3$, $y(0) = 1$.
(b) What is the interval on which the solution to part (a) is defined?
(c) Sketch to solution to part (a).

$$\frac{dy}{y^3} = dt$$

$$\int y^{-3}\,dy = \int dt$$

$$\frac{y^{-2}}{-2} = t + C$$

Find C now:

$$\frac{1^{-2}}{-2} = 0 + C$$

$$-\frac{1}{2} = C$$

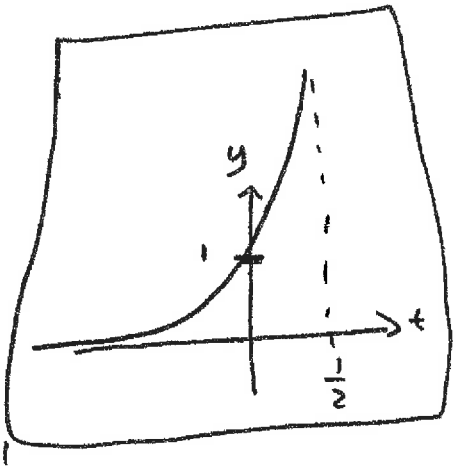
So $$\frac{y^{-2}}{-2} = t - \frac{1}{2}$$

$$y^{-2} = -2t + 1$$

$$\frac{1}{y^2} = 1 - 2t$$

$$y^2 = \frac{1}{1-2t}$$

$$\boxed{y = \frac{1}{\sqrt{1-2t}}}$$



(b) soln exists for $1-2t > 0$ or $2t < 1$
Solution defined for $\boxed{-\infty < t < \frac{1}{2}}$
or $\boxed{(-\infty, \frac{1}{2})}$

7. In this problem we will model pollution of a lake. Assume that the volume of the lake is 10^6 cubic meters. Initially the water in the lake is pure, but at time $t = 0$ a trans fat factory starts polluting the lake by pumping in effluent with a concentration of 2 kilograms per cubic meter of "substance F". The effluent is pumped in at a rate of 100 cubic meters per day. There is also a source of pure water entering the lake at a rate of 900 cubic meters per day. The pollutant in the lake is well mixed, and the mixed water drains out of the lake at the rate of 1000 cubic meters per day. (Thus, the volume of the lake stays at a constant 10^6 cubic meters.)

(a) Write down the IVP for $y(t)$, the kilograms of substance F in the lake after the factory has been open t days.
(b) Solve the IVP. (You may solve by inspection.)
(c) How many kilograms of substance F are in the lake in the limit $t \to \infty$?

$\frac{dy}{dt}$ = rate in − rate out $\left(\frac{kg}{day}\right)$

rate in = $2\frac{kg}{m^3} \cdot 100\frac{m^3}{day} = 200\frac{kg}{day}$

rate out = $\frac{y}{10^6}\frac{kg}{m^3} \cdot 10^3\frac{m^3}{day} = \frac{y}{10^3}\frac{kg}{day}$

$y(0) = 0$

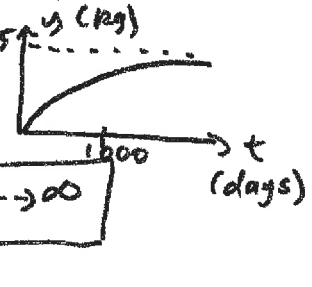
$$\frac{dy}{dt} = 200 - \frac{y}{10^3} = \frac{-y}{10^3} + \frac{10^3}{10^3} \cdot 200$$

$$= \frac{-1}{10^3}(y - 2\times10^5)$$

$$y = 2\times10^5 + C\,e^{-\frac{t}{10^3}}$$

$y(0) = 0$, so

$$y = 2\times10^5\left(1 - e^{-\frac{t}{10^3}}\right)$$



2×10^5 kg of substance F as $t \to \infty$

8. Solve the IVP $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

$$k^2 + 3k + 2 = 0$$

$$(k+1)(k+2) = 0$$

$$k_1 = -1 \quad k_2 = -2$$

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

$$y' = -c_1 e^{-t} - 2c_2 e^{-2t}$$

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = -c_1 - 2c_2 = 1$$

add $\quad 0 + -c_2 = 1$

$$c_2 = -1$$

~~$c_1 + c_2 = 0$~~

~~$y = -e^{-t} + e^{-2t}$~~

add $\quad 0 - c_2 = 1$

$$c_2 = -1$$

so $c_1 = 1$

$$\boxed{y = e^{-t} - e^{-2t}}$$

9. This problem concerns a linear nonhomogeneous ODE $L[y] = g(t)$.
Suppose that:
$y_1(t)$ satisfies the ODE with the initial conditions $y(0) = 1$, $y'(0) = 0$,
$y_2(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 1$, and
$y_3(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 0$.
Find the solution to the ODE with the initial conditions $y(0) = 3$, $y'(0) = 4$.
(Your answer will be a linear combination of y_1, y_2, and y_3.)

$L[c_1y_1 + c_2y_2 + c_3y_3] = (c_1 + c_2 + c_3)g(t)$

So $c_1 + c_2 + c_3 = 1$ to solve the ODE.

$c_1 = 3$ and $c_2 = 4$ to solve the IC. So $c_3 = -6$

$$\boxed{y = 3y_1 + 4y_2 - 6y_3}$$

10. Find the general solution to $x^2y'' - xy' - 3y = x^2$.

$y = y_h + Y$.

y_h solves $x^2y'' - xy' - 3y = 0$

try $y_h = x^r$

$r(r-1) - r - 3 = 0$

$r^2 - 2r - 3 = 0$

$(r+1)(r-3) = 0$

$r_1 = -1$ $r_2 = 3$

$y_h = c_1x^{-1} + c_2x^3$

Y has the form $Y = Ax^2$

$Y' = 2Ax$

$Y'' = 2A$

$x^2(2A) - x(2Ax) - 3Ax^2 = x^2$

$A = -\frac{1}{3}$ $Y = -\frac{1}{3}x^2$

$$\boxed{y = c_1x^{-1} + c_2x^3 - \frac{1}{3}x^2}$$

NOT assigned

13. Find the general solution to $\frac{dx}{dt} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x$.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ You might know that the evals are $\lambda = 2 \pm 1$.

$\det(A - \lambda I) = 0$ is $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 - 1 = 0$ OR $(2-\lambda)^2 - 1 = 0$

$4 - 4\lambda + \lambda^2 - 1 = 0$ $(2-\lambda)^2 = 1$

$\lambda^2 - 4\lambda + 3 = 0$ $2 - \lambda = \pm 1$

$(\lambda - 1)(\lambda - 3) = 0$ $\lambda = 2 \pm 1$.

$\lambda_1 = 1$, $\lambda_2 = 3$ (or the other way around.).

$(A - \lambda_1 I)\vec{v}^{(1)} = \vec{0}$ is $(A - I)\vec{v}^{(1)} = \vec{0}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}^{(1)} = \vec{0}$$

one choice is $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$(A - \lambda_2 I)\vec{v}^{(2)} = \vec{0}$ is $(A - 3I)\vec{v}^{(2)} = \vec{0}$

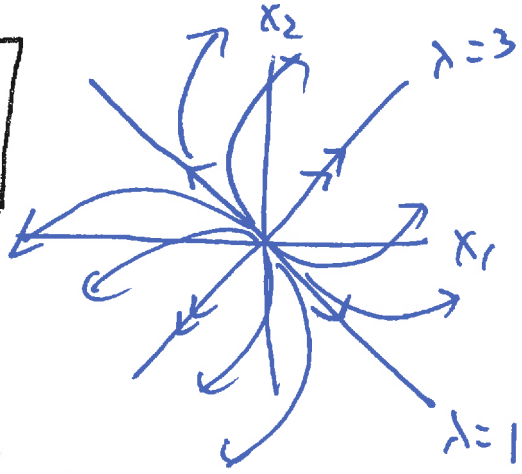
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v}^{(2)} = \vec{0}$$

one choice is $\vec{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, the general solution is

$$\boxed{\vec{x} = c_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Phase portrait



14. Find the general solution to $\frac{dx}{dt} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x$.

spirals clockwise, since $\dot{x}_1 = \cdots + 2x_2$ ($+2x_2$ positive)

You might know that the eigenvalues are $\lambda = 1 \pm 2i$. otherwise calculate them.

spirals out, since $\mathrm{tr}(A) > 0$

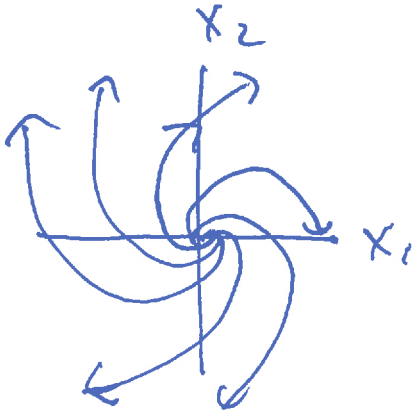
$\det(A - \lambda I) = 0$ is $\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0$

$\lambda^2 - 2\lambda + 5 = 0$; or $(1-\lambda)^2 + 4 = 0$

$\lambda = \frac{+2 \pm \sqrt{4-20}}{2}$ $\quad (1-\lambda)^2 = -4$

$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$ $\quad 1-\lambda = \pm 2i$

$\lambda = 1 \pm 2i$.



phase portrait

Find evector of $\lambda_1 = 1+2i$ ($\alpha = 1$, $\beta = 2$)

$(A - \lambda_1 I)\vec{v}^{(1)} = \vec{0}$ is $\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \vec{v}^{(1)} = 0$

$\vec{v}^{(1)} = \begin{bmatrix} 2 \\ 2i \end{bmatrix}$ or $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ or ...

use $\vec{v}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

so $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the formula.

$\vec{u} = e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) = e^{t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(2t)\right)$

$\vec{u} = e^{t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$; $\vec{v} = e^{t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t)\right)$

$$\vec{x} = c_1 e^{t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 e^{t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

Extra Credit: Draw a cartoon of General Solution.

up to 2 points (before scaling)



(15) $ay'' + by' + cy = 0$

char eqn. $a\lambda^2 + b\lambda + c = 0$.

(a) Let $x_1 = y$, $x_2 = y'$. Therefore $x_1' = x_2$

$$x_2' = y'' = \frac{-by' - cy}{a} = -\frac{b}{a}x_2 - \frac{c}{a}x_1$$

So $x_1' = x_2$

$x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2$, or $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}\vec{x}$

That is,

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}.$$

(b) The eigenvalues of A satisfy the char. equation

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{bmatrix} = -\lambda\left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = 0$$

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$$

we can mult. through by a, since $a \neq 0$ and get

$$\boxed{a\lambda^2 + b\lambda + c = 0}$$ Same equation as before!

So the eigenvalues of A are the same as the roots of the char. eqn. of the ODE.