Perko Section 1.1

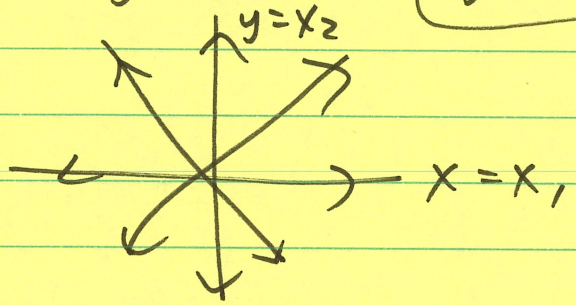
1 Find the general solution & draw the phase portrait:

(a) $\dot{x} = x$
$\dot{y} = y$

$x = c_1 e^t$
$y = c_2 e^t$

$\therefore \quad y = \left(\frac{c_2}{c_1}\right) x = cx$

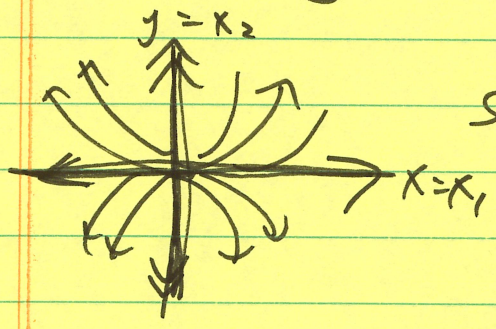
solutions are lines through origin.



(c) $\dot{x} = x$
$\dot{y} = 2y$

$x = c_1 e^t$
$y = c_2 e^{2t} = c_2 \left(\frac{x}{c_1}\right)^2 = \frac{c_2}{c_1^2} x^2$
$y = cx^2$

solutions lie on parabolas (or are on the y-axis)



(d) $\dot{x} = -y \quad \therefore \quad \ddot{x} = -\dot{y} = -(x), \quad \ddot{x} = -x$
$\dot{y} = x$

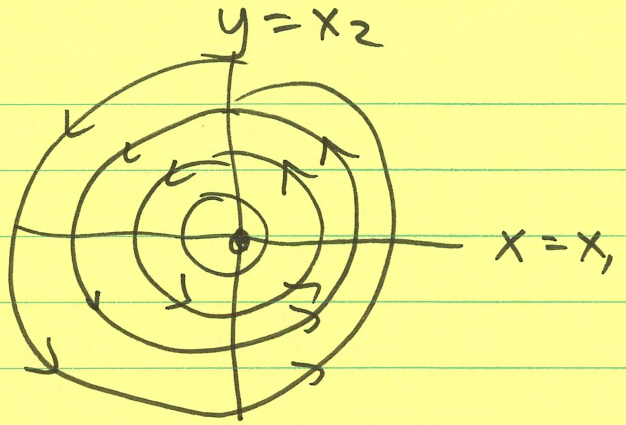
so

$x = c_1 \cos(t) + c_2 \sin(t)$
$y = c_1 \sin(t) - c_2 \cos(t)$

differentiate to get

$\frac{d}{dt}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = 2x(-y) + 2yx = -2xy + 2xy = 0$. so $x^2 + y^2 = \text{const}$. Solutions lie on circles centered at the origin

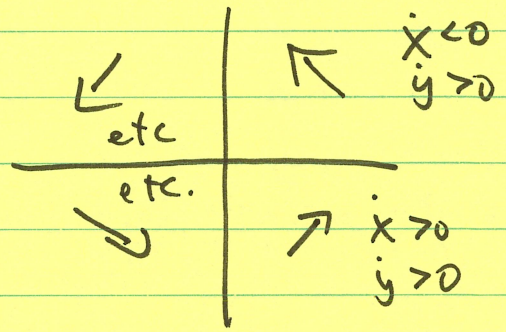
§1.1 p.2



(d) continued

look at signs of $\dot{x}$ and $\dot{y}$ to see direction

$\dot{x} = -y$
$\dot{y} = x$ } so



Note: using the general solution to show $(x(t))^2 + (y(t))^2 = \text{constant}\ (= c_1^2 + c_2^2)$ is an OK way to do this, but ~~Painful.~~ Painful.

(e) $\dot{x} = -x + y$
$\dot{y} = -y$ $\quad \therefore y = c_2 e^{-t}$

Now $\dot{x} + x = c_2 e^{-t}$ is linear

$\mu(t) = e^t$

$e^t \dot{x} + e^t x = c_2 e^t e^{-t} = c_2$

$\frac{d}{dt}(e^t x) = c_2$, $e^t x = c_2 t + c_1$

$$\boxed{\begin{aligned} x(t) &= c_1 e^{-t} + c_2 t e^{-t} \\ y(t) &= c_2 e^{-t} \end{aligned}}$$ general solution

(e) continued. To eliminate t,

use $y = C_2 e^{-t}$, so $-t = \ln\left(\frac{y}{C_2}\right)$

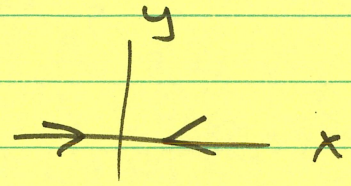
$t = -\ln\left(\frac{y}{C_2}\right)$

$x = C_1 e^{-t} + C_2 t e^{-t}$ is

$x = C_1\left(\frac{y}{C_2}\right) + C_2\left(-\ln\left(\frac{y}{C_2}\right)\right)\frac{y}{C_2}$

$x = \left(\frac{C_1}{C_2}\right) y - \ln\left(\frac{y}{C_2}\right) y$

Case I. $C_2 = 0$: $\boxed{\begin{array}{l} x(t) = C_1 e^{-t} \\ y(t) = 0 \end{array}}$



Case II $C_2 > 0$ (so $y > 0$) $x = \frac{C_1}{C_2} y - \left(\ln(y) - \ln($

$x = \left(\frac{C_1}{C_2}\right) + \ln(C_2)\, y - \ln(y)$

$\boxed{x = \tilde{c}\, y - \ln(y)\, y}$ where $\tilde{c} = \frac{C_1}{C_2} + \ln(C_2)$, an arbitrary const.

Case III $C_2 < 0$, so $y < 0$.

$x = \frac{C_1}{C_2} y - \left(\ln(-y) - \ln(-C_2)\right) y$

$x = \left(\frac{C_1}{C_2} + \ln(-C_2)\right) y - \ln(-y)\, y = \tilde{c}\, y - \ln(-y)\, y$

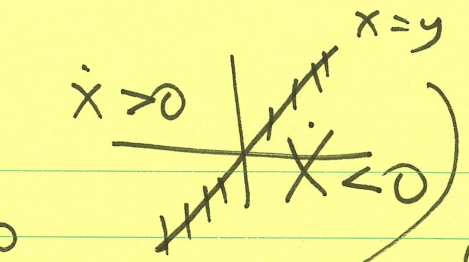
these can be combined to ~~$y = \tilde{c}$~~ $\boxed{x = \tilde{c} y - \ln|y|\, y}$

~~here~~ $\boxed{x = cy - \ln|y|\, y}$

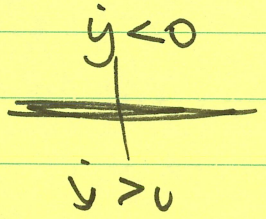
This can, but looking at the signs of $\dot{x}$ and $\dot{y}$ helps!

§1.1 p4

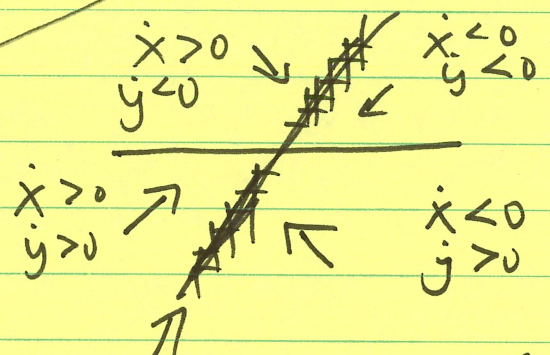
$\dot{x} = -x + y$



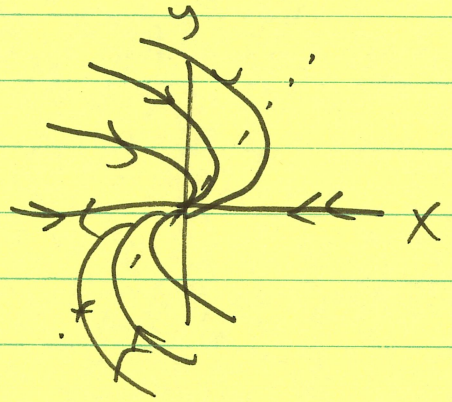
$\dot{y} = -y$



combines to



ẋ=0 on this diagonal trajectories have a vertical tangent



this can be graphed with desmos

$x = cy - \ln(|y|)y$, or $y = 0$

$c_2 \neq 0$ $\quad$ $c_2 = 0$

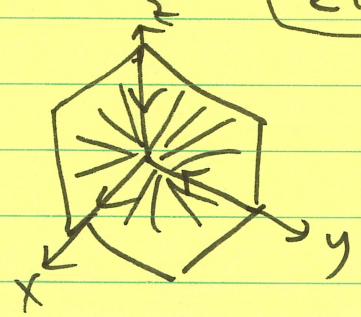
Plotting $y = x$ too, and $y = 0$ $(y = 0 \cdot x)$ in blue
green dotted is nice ↑ this avoids a slider.

The website has a link to these graphs.

2a. $\dot{x} = x$
$\dot{y} = y$
$\dot{z} = z$

$x(t) = c_1 e^t$
$y(t) = c_2 e^t$
$z(t) = c_3 e^t$

all solutions line on rays from the origin



The phase portrait is hard to draw. Note: This is Hubble's law for the expanding universe!

from 16a/ §6.1 p5

$\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = -z$

$x(t) = C_1\cos(t) + C_2\sin(t)$

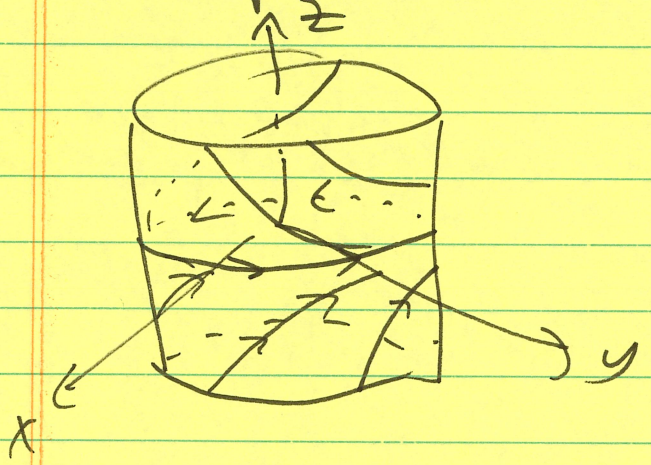
$y(t) = C_1\sin(t) - C_2\cos(t)$

$z(t) = C_3 e^{-t}$

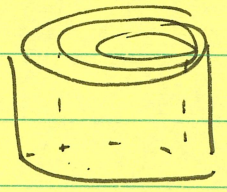
(+trajectories)

Solutions lie on the cylinder $x^2+y^2 = C^2$ in $\mathbb{R}^3$

The solutions spin counterclockwise when viewed from above, and approach the x-y plane exponentially. Here's an attempt at a drawing.



A Desmos 3D graph is saved.



phase portrait has nested cylinders.

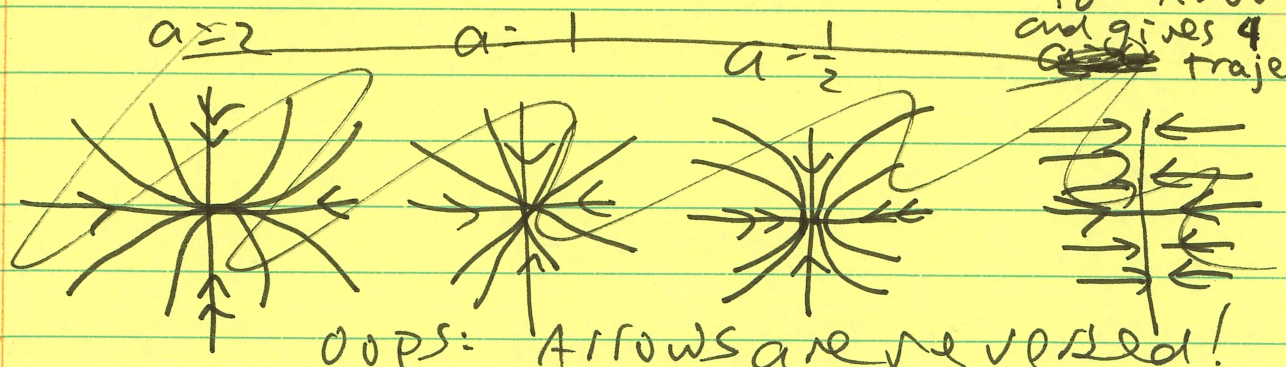
3. $\dot{x} = x$, $\dot{y} = ay$

$x = C_1 e^t$

$y = C_2 e^{at}$ = eliminate t $= C_2 (e^t)^a$

$y = \left(\frac{C_2}{C_1^a}\right) x^a$ $\boxed{y = cx^a}$ for $x>0$ this is good

$\boxed{|y| = c|x|^a}$ for $x \neq 0$, this works for $x>0$ or $x<0$, and gives 4 trajectories

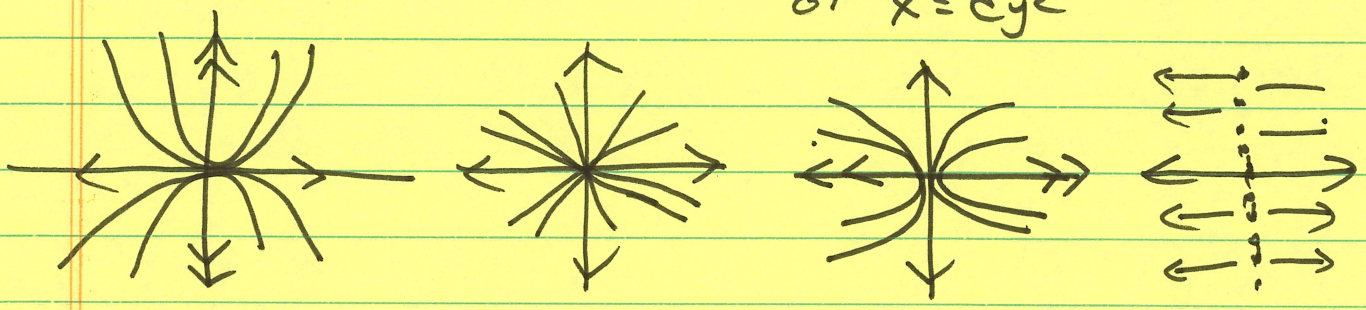


oops: Arrows are reversed!

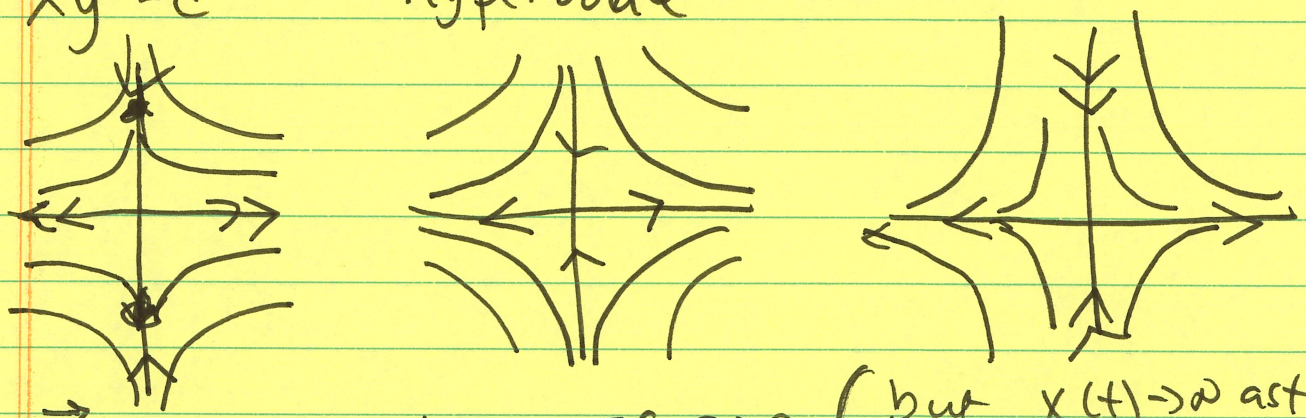
3 continued

Starting with $a>0$ is easiest for me

$a=2$ $y=cx^2$ | $a=1$ $y=cx$ | $a=\frac{1}{2}$ $y=c|x|^{\frac{1}{2}}$ or $x=cy^2$ | $a=0$ $y=\text{const}$



$a=-\frac{1}{2}$ $y=c|x|^{-\frac{1}{2}}$ $xy^2=\tilde{c}$ | $a=-1$ $xy=\tilde{c}$ hyperbolae | $a=-2$ $y=cx^{-2}$ $x^2y=c$



$\vec{x}(t)\to 0$ as $t\to-\infty$ if $a>0$

(but $x(t)\to\infty$ as $t\to\infty$ and $y(t)\to 0$ as $t\to\infty$ if $a<0$)

4. $A=\text{diag}[\lambda_1,\lambda_2,\ldots\lambda_n]$

$x_i(t)=c_i e^{\lambda_i t}$ general solution.

($x_i(t)\to 0$ for all i as $t\to\infty$) if and only if ($\lambda_i<0$ for all i)

This is equivalent to $\vec{x}(t)\to\vec{0}$ as $t\to\infty$, or $\lim_{t\to\infty}\vec{x}(t)=\vec{0}$.

§7.1 p.7

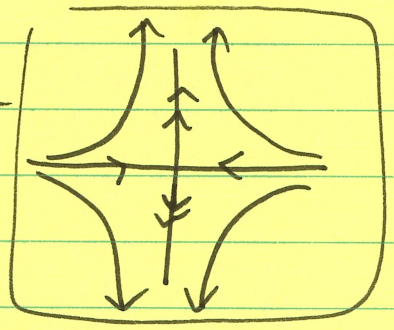
5. The vector fields $\vec{F}_1(\vec{x}) = A\vec{x}$ and $\vec{F}_2(\vec{x}) = kA\vec{x}$ point in the same direction if $k>0$, and they point in the opposite direction if $k<0$. The phase portrait is the same for all $k>0$, and also the same for all $k<0$

Bonus 1. $\frac{dx}{dt} = -x \qquad x(t) = c_1 e^{-t}$

$\frac{dy}{dt} = 2y \qquad y(t) = c_2 e^{2t} = c_2 (e^{-t})^{-2}$

$y = c_2 \left(\frac{x}{c_1}\right)^{-2} = \frac{(c_2 c_1^2)}{x^2} = \frac{c}{x^2}$

$y = \frac{c}{x^2}$

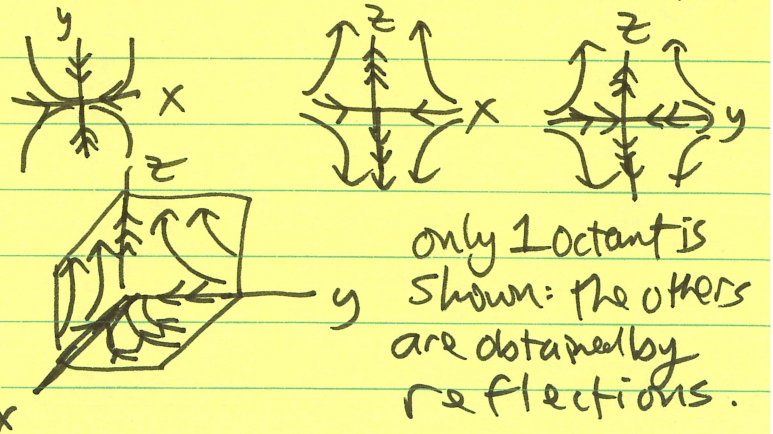


Bonus 2. $\frac{dx}{dt} = -x$

$\frac{dy}{dt} = -2y$

$\frac{dz}{dt} = 3z$

Sketch the 3 coordinate planes



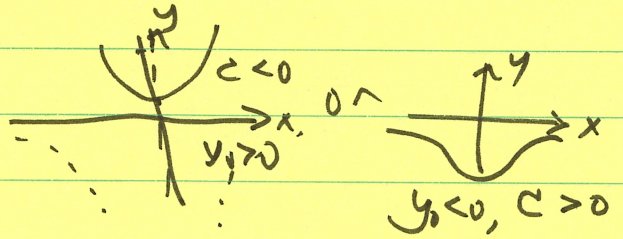
only 1 octant is shown: the others are obtained by reflections.

Bonus 3. Find the general solution to $\frac{dy}{dx} = 2xy^2$. (Every C gives a solution & every solution, including it's interval of existence, is obtained for some C.

Standard separation of variables gives $y(x) = \frac{-1}{C+x^2}$. I will define 3 families of solutions that can be sketched together.

$y(x) = \frac{y_0}{1-y_0x^2}$, defined on $\{x \in \mathbb{R} \mid 1-y_0x^2 > 0\}$

satisfies $y(0) = y_0$



For $x_0 \leq -1$ we have

$y = \frac{1}{x_0^2 - x^2 - 1}$, on $\{x \in \mathbb{R} \mid x < -\sqrt{x_0^2-1}\}$

$y(x_0) = -1$ for both of these

For $x_0 \geq -1$, we have $y = \frac{1}{x_0^2 - x^2 - 1}$, for $x > \sqrt{x_0^2-1}$

Thus we can get the general solution

$$y(x) = \begin{cases} \frac{1}{C^2-x^2-1} \text{ on } (-\infty, -\sqrt{C^2-1}) & \text{for } C \leq -1 \quad (\text{here } C = x_0) \\ \frac{\tanh^{-1}(C)}{1-\tanh^{-1}(C)x^2} \text{ on } \{x \in \mathbb{R} \mid 1-\tanh^{-1}(C)x^2 > 0\} & \text{for } -1 < C < 1 \quad (\text{here } y_0 = \tanh^{-1}(C)) \\ \frac{1}{C^2-x^2-1} \text{ on } (\sqrt{C^2-1}, \infty) & \text{for } C \geq 1 \quad \text{here, } C = x_0 \end{cases}$$

See the desmos graph.